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Regular Elements of the Complete Semigroups $B_X(D)$ of Binary Relations of the Class $\Sigma_2(X,8)$

Nino Tsinaridze, Shota Makharadze

Department of Mathematics, Faculty of Mathematics, Physics and Computer Sciences, Shota Rustaveli Batumi State University, Batumi, Georgia
Email: ninocinaridze@mail.ru, shota_59@mail.ru

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Abstract

As we know if D is a complete X-semilattice of unions then semigroup $B_X(D)$ possesses a right unit iff D is an XI-semilattice of unions. The investigation of those α-idempotent and regular elements of semigroups $B_X(D)$ requires an investigation of XI-subsemilattices of semilattice D for which $V(D,\alpha)=Q\in\Sigma_2(X,8)$. Because the semilattice Q of the class $\Sigma_2(X,8)$ are not always XI-semilattices, there is a need of full description for those idempotent and regular elements when $V(D,\alpha)=Q$. For the case where X is a finite set we derive formulas by calculating the numbers of such regular elements and right units for which $V(D,\alpha)=Q$.

Keywords

Semilattice, Semigroup, Binary Relation

1. Introduction

In this paper we characterize the elements of the class $\Sigma_2(X,8)$. This class is the complete X-semilattice of unions every elements of which are isomorphic to Q. So, we characterize the class for each element which is isomorphic to Q by means of the characteristic family of sets, the characteristic mapping and the generate set of D.

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Let X be an arbitrary nonempty set, recall that the set of all binary relations on X is denoted B_X. The binary operation "$\circ$" on B_X defined by for $\alpha, \beta \in B_X$ $(x,z) \in \alpha \circ \beta \Leftrightarrow (x,y) \in \alpha$ and $(y,z) \in \beta$, for some $y \in X$ is associative and hence B_X is a semigroup with respect to the operation "$\circ$". This semigroup is called the semigroup of all binary relations on the set X. By $\varnothing$ we denote an empty binary relation or empty subset of the set X.

Let D be a X-semilattice of unions, *i.e.* a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there corresponds a binary relation α_f on the set X that satisfies the condition $\alpha_f = \bigcup(\{x\} \times f(x))$. The set of all such α_f $(f : X \to D)$ is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by a X-semilattice of unions D (see ([1], Item 2.1), ([2], Item 2.1)).

Let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\varnothing \neq D' \subseteq D$ and $t \in \breve{D} = \bigcup_{Y \in D} Y$. We use the notations:

$$y\alpha = \{x \in X \mid y\alpha x\},\ Y\alpha = \bigcup_{y \in Y} y\alpha,\ V(D,\alpha) = \{Y\alpha \mid Y \in D\},\ X^* = \{T \mid \varnothing \neq T \subseteq X\},$$

$$l(D',T) = \cup(D' \setminus D'_T),\ Y_T^\alpha = \{x \in X \mid x\alpha = T\}\, D'_t = \{Z' \in D' \mid t \in Z'\},$$

$$D'_T = \{Z' \in D' \mid T \subseteq Z'\},\ \ddot{D}'_T = \{Z' \in D' \mid Z' \subseteq T\}.$$

Let $\alpha \in B_X(D)$, $Y_T^\alpha = \{x \in X \mid x\alpha = T\}$ and

$$V[\alpha] = \begin{cases} V(X^*,\alpha), & \text{if } \varnothing \notin D; \\ V(X^*,\alpha), & \text{if } \varnothing \in V(X^*,\alpha); \\ V(X^*,\alpha) \cup \{\varnothing\}, & \text{if } \varnothing \notin V(X^*,\alpha) \text{ and } \varnothing \in D. \end{cases}$$

In general, a representation of a binary relation α of the form $\alpha = \bigcup_{T \in V[\alpha]} (Y_T^\alpha \times T)$ is called quasinormal.

Note that for a quasinormal representation of a binary relation α, not all sets Y_T^α $(T \in V[\alpha])$ can be different from an empty set. But for this representation the following conditions are always fulfilled:

a) $Y_T^\alpha \cap Y_{T'}^\alpha = \varnothing$, for any $T, T' \in D$ and $T \neq T'$;

b) $X = \bigcup_{T \in V[\alpha]} Y_T^\alpha$ (see ([1], Definition 1.11.1), ([2], Definition 1.11.1)).

Let $\varepsilon \in B_X(D)$. ε is called right unit of the semigroup $B_X(D)$. If $\alpha \circ \varepsilon = \alpha$ for any $\alpha \in B_X(D)$. An element α taken from the semigroup $B_X(D)$ called a regular element of the semigroup $B_X(D)$ if in $B_X(D)$ there exists an element β such that $\alpha \circ \beta \circ \alpha = \alpha$ (see [1]-[3]).

In [1] [2] they show that β is regular element of $B_X(D)$ iff $V[\beta] = V(D,\beta)$ is a complete XI-semilattice of unions.

A complete X-emilattice of unions D is an XI-emilattice of unions if it satisfies the following two conditions:

(a) $\wedge(D, D_t) \in D$ for any $t \in \breve{D}$;

(b) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty element Z of D (see ([1], Definition 1.14.2), ([2], Definition 1.14.2) or [4]). Under the symbol $\wedge(D, D_t)$ we mean an exact lower bound of the set D_t in the semilattice D.

Let D' be an arbitrary nonempty subset of the complete X-semilattice of unions D. A nonempty element T is a nonlimiting element of the set D' if $T \setminus l(D',T) \neq \varnothing$ and a nonempty element T is a limiting element of the set D' if $T \setminus l(D',T) = \varnothing$ (see ([1], Definition 1.13.1 and Definition 1.13.2), ([2], Definition 1.13.1 and Definition 1.13.2)).

Let $D = \{\breve{D}, Z_1, Z_2, \cdots, Z_{m-1}\}$ be some finite X-semilattice of unions and $C(D) = \{P_0, P_1, P_2, \cdots, P_{m-1}\}$ be the family of sets of pairwise nonintersecting subsets of the set X. If φ is a mapping of the semilattice D on the family of sets $C(D)$ which satisfies the condition $\varphi(\breve{D}) = P_0$ and $\varphi(Z_i) = P_i$ for any $i = 1, 2, \cdots, m-1$ and $\hat{D}_Z = D \setminus \{T \in D | Z \subseteq T\}$, then the following equalities are valid:

$$\breve{D} = P_0 \cup P_1 \cup P_2 \cup \cdots \cup P_{m-1}, \quad Z_i = P_0 \cup \bigcup_{T \in \hat{D}_{Z_i}} \varphi(T) \tag{$\bullet$}$$

In the sequel these equalities will be called formal.

It is proved that if the elements of the semilattice D are represented in the form $(\bullet)$, then among the parameters P_i $(i = 0, 1, 2, \cdots, m-1)$ there exist such parameters that cannot be empty sets for D. Such sets P_i $(0 < i \le m-1)$ are called basis sources, whereas sets P_j $(0 \le j \le m-1)$ which can be empty sets too are called completeness sources.

It is proved that under the mapping φ the number of covering elements of the pre-image of a basis source is always equal to one, while under the mapping φ the number of covering elements of the pre-image of a completeness source either does not exist or is always greater than one (see ([1], Item 11.4), ([2], Item 11.4) or [5]).

The one-to-one mapping φ between the complete X-semilattices of unions $\phi(Q,Q)$ and D'' is called a complete isomorphism if the condition

$$\varphi(\cup D_1) = \bigcup_{T \in D_1} \varphi(T')$$

Is fulfilled for each nonempty subset D_1 of the semilattice D' (see ([1], definition 6.3.2), ([2], definition 6.3.2) or [6]) and the complete isomorphism φ between the complete semilattices of unions Q and D' is a complete α-isomorphism if (b)

(a) $Q = V(D,\alpha)$;

(b) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D,\alpha)$ and $\varphi(T)\alpha = T$ for all $T \in V(D,\alpha)$ (see ([1], Definition 6.3.3), ([2], Definition 6.3.3)).

Lemma 1.1. Let D by a complete X-semilattice of unions. If a binary relation ε of the form $\varepsilon = \bigcup_{t \in \breve{D}} (\{t\} \times \wedge(D, D_t)) \cup ((X \setminus \breve{D}) \times \breve{D})$ is right unit of the semigroup $B_X(D)$, then ε is the greatest right unit of that semigroup (see ([1], Lemma 12.1.2), ([2], Lemma 12.1.2)).

Theorem 1.1. Let $D_j = \{T_1, T_2, \cdots, T_j\}$, X and Y—be three such sets, that $\varnothing \ne Y \subseteq X$. If f is such mapping of the set X, in the set D_j, for which $f(y) = T_j$ for some $y \in Y$, then the numbers of all those mappings f of the set X in the set D_j is equal to $s = j^{|X \setminus Y|} \cdot \left(j^{|Y|} - (j-1)^{|Y|}\right)$ (see ([1], Theorem 1.18.2), ([2], Theorem 1.18.2)).

Theorem 1.2. Let D be a finite X-semilattice of unions and $\alpha \circ \sigma \circ \alpha = \alpha$ for some α and σ of the semigroup $B_X(D)$; $D(\alpha)$ be the set of those elements T of the semilattice $Q = B_X(D) \setminus \{\varnothing\}$ which are nonlimiting elements of the set $\ddot{Q}_T$. Then a binary relation α having a quasinormal representation of the form $\alpha = \bigcup_{T \in V(D,\alpha)} Y_T^\alpha \times T$ is a regular element of the semigroup $B_X(D)$ iff the set $V(D,\alpha)$ is a XI-semilattice of unions and for α-isomorphism φ of the semilattice $V(D,\alpha)$ on some X-subsemilattice D' of the semilattice D the following conditions are fulfilled:

(a) $\varphi(T) = T\sigma$ for any $T \in V(D,\alpha)$;

(b) $\bigcup_{T \in \ddot{D}(\alpha)_T} Y_T^\alpha \supseteq \varphi(T)$ for any $T \in D(\alpha)$;

(c) $Y_T^\alpha \cap \varphi(T) \ne \varnothing$ for any element T of the set $\ddot{D}(\alpha)_T$ (see ([1], Theorem 6.3.3), ([2], Theorem 6.3.3) or [6]).

Theorem 1.3. Let D be a complete X-emilattice of unions. The semigroup $B_X(D)$ possesses a right unit iff D is an XI-semilattice of unions (see ([1], Theorem 6.1.3), ([2], Theorem 6.1.3) or [7]).

2. Results

Let D is any X -semilattice of unions and $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\subseteq D$, which satisfies the following conditions:

$$\begin{aligned}
&T_7\subset T_5\subset T_2\subset T_0,\ \ T_7\subset T_4\subset T_2\subset T_0,\ \ T_7\subset T_4\subset T_1\subset T_0,\\
&T_6\subset T_4\subset T_2\subset T_0,\ \ T_6\subset T_4\subset T_1\subset T_0,\ \ T_6\subset T_3\subset T_1\subset T_0,\\
&T_7\cup T_6=T_4,\ \ T_5\cup T_4=T_2,\ \ T_4\cup T_3=T_1,\ \ T_2\cup T_1=T_0,\\
&T_1\setminus T_2\neq\varnothing,\ \ T_2\setminus T_1\neq\varnothing,\ \ T_3\setminus T_4\neq\varnothing,\ \ T_4\setminus T_3\neq\varnothing,\\
&T_3\setminus T_5\neq\varnothing,\ \ T_5\setminus T_3\neq\varnothing,\ \ T_4\setminus T_5\neq\varnothing,\ \ T_5\setminus T_4\neq\varnothing,\\
&T_6\setminus T_7\neq\varnothing,\ \ T_7\setminus T_6\neq\varnothing.
\end{aligned}\tag{1}$$

The semilattice Q, which satisfying the conditions (1) is shown in **Figure 1**. By the symbol $\Sigma_2(X,8)$ we denote the set of all X -semilattices of unions whose every element is isomorphic to Q.

Let $C(Q)=\{P_7,P_6,P_5,P_4,P_3,P_2,P_1,P_0\}$ is a family sets, where $P_7,\ P_6,\ P_5,\ P_4,\ P_3,\ P_2,\ P_1,\ P_0$ are pairwise disjoint subsets of the set X and

$$\psi=\begin{pmatrix} T_7 & T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ P_7 & P_6 & P_5 & P_4 & P_3 & P_2 & P_1 & P_0\end{pmatrix}$$

is a mapping of the semilattice Q into the family sets $C(Q)$. Then for the formal equalities of the semilattice Q we have a form:

$$\begin{aligned}
T_0&=P_0\cup P_1\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
T_1&=P_0\cup P_2\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
T_2&=P_0\cup P_1\cup P_3\cup P_4\cup P_5\cup P_6\cup P_7,\\
T_3&=P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7,\\
T_4&=P_0\cup P_3\cup P_5\cup P_6\cup P_7,\\
T_5&=P_0\cup P_1\cup P_3\cup P_4\cup P_6\cup P_7,\\
T_6&=P_0\cup P_5\cup P_7,\\
T_7&=P_0\cup P_3\cup P_6.
\end{aligned}\tag{2}$$

here the elements $P_1,\ P_2,\ P_3,\ P_5$ are basis sources, the element $P_0,\ P_4,\ P_6,\ P_7$ are sources of completenes of the semilattice Q. Therefore $|X|\geq 4$ and $\delta=4$.

Theorem 2.1. Let $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\in\Sigma_2(X,8)$. Then Q is XI -semilattice, when $T_5\cap T_3=\varnothing$.

Proof. Let $t\in T_0$, $Q_t=\{T\in Q\,|\,t\in T\}$ and $\wedge(Q,Q_t)$ is the exact lower bound of the set Q_t in Q. Then of the formal equalities (2) follows, that

$$Q_t=\begin{cases}
Q, & \text{if } t\in P_0,\\
\{T_5,T_2,T_0\}, & \text{if } t\in P_1,\\
\{T_3,T_1,T_0\}, & \text{if } t\in P_2,\\
\{T_7,T_5,T_4,T_2,T_1,T_0\}, & \text{if } t\in P_3,\\
\{T_5,T_3,T_2,T_1,T_0\}, & \text{if } t\in P_4,\\
\{T_6,T_4,T_3,T_2,T_1,T_0\}, & \text{if } t\in P_5,\\
\{T_7,T_5,T_4,T_3,T_2,T_1,T_0\}, & \text{if } t\in P_6,\\
\{T_6,T_5,T_4,T_3,T_2,T_1,T_0\}, & \text{if } t\in P_7,
\end{cases}
\qquad
\wedge(Q,Q_t)=\begin{cases}
T_5, & \text{if } t\in P_1,\\
T_3, & \text{if } t\in P_2,\\
T_7, & \text{if } t\in P_3,\\
T_6, & \text{if } t\in P_5,
\end{cases}$$

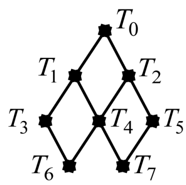

Figure 1. Diagram of Q.

We have $Q^{\wedge}=\{\wedge(Q,Q_t)\,|\,t\in T_0\}=\{T_7,T_6,T_5,T_3\}$ and $\wedge(Q,Q_t)\notin Q$ if $t\in P_0\cup P_4\cup P_6\cup P_7$. So, from the definition XI-semilattice follows that Q is not XI-semilattice.

If $P_0=P_4=P_6=P_7=\varnothing$ (since they are completeness sources), then $\wedge(Q,Q_t)\in Q$ for all $t\in T_0$ and $T_4=T_7\cup T_6$, $T_1=T_7\cup T_3$, $T_2=T_6\cup T_5$. Of the last conditions and from the Definition XI-semilattice follows that Q is XI-semilattice. Of the equality $P_0=P_4=P_6=P_7=\varnothing$ follows that

$$T_5\cap T_3=(P_0\cup P_1\cup P_3\cup P_4\cup P_6\cup P_7)\cap(P_0\cup P_2\cup P_4\cup P_5\cup P_6\cup P_7)=P_0\cup P_4\cup P_6\cup P_7=\varnothing$$

Of the other hand, if $T_5\cap T_3=\varnothing$ then by formal equalities follows that $P_0=P_4=P_6=P_7=\varnothing$. Therefore, semilattice Q is XI-semilattice.

The Theorem is proved.

Lemma 2.1. Let $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\in\Sigma_2(X,8)$ and $T_5\cap T_3=\varnothing$. Then following equalities are true:

$$P_3=T_7,\ P_5=T_6,\ P_2=T_3\setminus T_2,\ P_1=T_5\setminus T_1$$

Proof. The given Lemma immediately follows from the formal equalities (2) of the semilattice Q.

The lemma is proved.

Lemma 2.2. Let $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\in\Sigma_2(X,8)$ and $T_5\cap T_3=\varnothing$. Then the binary relation

$$\varepsilon=(T_7\times T_7)\cup(T_6\times T_6)\cup((T_5\setminus T_1)\times T_5)\cup((T_3\setminus T_2)\times T_3)\cup((X\setminus T_0)\times T_0)$$

is the largest right unit of the semigroup $B_X(D)$.

Proof. By preposition and from Theorem 2.1 follows that Q is XI-semilattice. Of this, from Lemma 1.1, from Lemma 2.1 and from Theorem 1.3 we have that the binary relation

$$\begin{aligned}\varepsilon&=\bigcup_{t\in T_0}(\{t\}\times\wedge(Q,Q_t))\cup((X\setminus T_0)\times T_0)=(P_3\times T_7)\cup(P_5\times T_6)\cup(P_1\times T_5)\cup(P_2\times T_3)\cup((X\setminus T_0)\times T_0)\\&=(T_7\times T_7)\cup(T_6\times T_6)\cup((T_5\setminus T_1)\times T_5)\cup((T_3\setminus T_2)\times T_3)\cup((X\setminus T_0)\times T_0).\end{aligned}$$

is the largest right unit of the semigroup $B_X(D)$.

The lemma is proved.

Lemma 2.3. Let $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\in\Sigma_2(X,8)$ and $T_5\cap T_3=\varnothing$. Binary relation α having quazi-normal representation of the form

$$\alpha=(Y_7^\alpha\times T_7)\cup(Y_6^\alpha\times T_6)\cup(Y_5^\alpha\times T_5)\cup(Y_4^\alpha\times T_4)\cup(Y_3^\alpha\times T_3)\cup(Y_2^\alpha\times T_2)\cup(Y_1^\alpha\times T_1)\cup(Y_0^\alpha\times T_0)$$

where Y_7^α, Y_6^α, Y_5^α, $Y_3^\alpha\notin\{\varnothing\}$ and $V(D,\alpha)=Q\in\Sigma_2(X,8)$ is a regular element of the semigroup $B_X(D)$ iff for some complete α isomorphism $\varphi=\begin{pmatrix}T_7 & T_6 & T_5 & T_4 & T_3 & T_2 & T_1 & T_0\\ P_7 & \overline{T}_6 & \overline{T}_5 & \overline{T}_4 & \overline{T}_3 & \overline{T}_2 & \overline{T}_1 & \overline{T}_0\end{pmatrix}$ of the semilattice Q on some X-subsemilattice $Q'=\{\overline{T}_7,\overline{T}_6,\overline{T}_5,\overline{T}_4,\overline{T}_3,\overline{T}_2,\overline{T}_1,\overline{T}_0\}$ (see **Figure 2**) of the semilattice Q satisfies the following conditions:

$$Y_7^\alpha\supseteq\overline{T}_7,\ Y_6^\alpha\supseteq\overline{T}_6,\ Y_7^\alpha\cup Y_5^\alpha\supseteq\overline{T}_5,\ Y_6^\alpha\cup Y_3^\alpha\supseteq\overline{T}_3,\ Y_5^\alpha\cap\overline{T}_5\neq\varnothing,\ Y_3^\alpha\cap\overline{T}_3\neq\varnothing$$

Proof. It is easy to see, that the set $Q(\alpha)=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1\}$ is a generating set of the semilattice Q. Then the following equalities are hold:

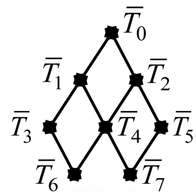

Figure 2. Diagram of Q'.

$$\ddot{Q}(\alpha)_{T_7} = \{T_7\},\ \ddot{Q}(\alpha)_{T_6} = \{T_6\},\ \ddot{Q}(\alpha)_{T_5} = \{T_7, T_5\},\ \ddot{Q}(\alpha)_{T_4} = \{T_7, T_6, T_4\},$$
$$\ddot{Q}(\alpha)_{T_3} = \{T_6, T_3\},\ \ddot{Q}(\alpha)_{T_2} = \{T_7, T_6, T_5, T_4, T_2\},\ \ddot{Q}(\alpha)_{T_1} = \{T_7, T_6, T_4, T_3, T_1\}.$$

By Statement b) of the Theorem 1.2 follows that the following conditions are true:

$$Y_7^\alpha \supseteq \overline{T}_7,\ Y_6^\alpha \supseteq \overline{T}_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq \overline{T}_5,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq \overline{T}_4,\ Y_6^\alpha \cup Y_3^\alpha \supseteq \overline{T}_3,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq \overline{T}_2,\ Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq \overline{T}_1;$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq \overline{T}_7 \cup \overline{T}_6 \cup Y_4^\alpha = \overline{T}_4 \cup Y_4^\alpha \supseteq \overline{T}_4,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha = \left(Y_7^\alpha \cup Y_5^\alpha\right) \cup \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup Y_2^\alpha \supseteq \overline{T}_5 \cup \overline{T}_4 \cup Y_2^\alpha = \overline{T}_2 \cup Y_2^\alpha \supseteq \overline{T}_2,$$
$$Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha = \left(Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha\right) \cup \left(Y_6^\alpha \cup Y_3^\alpha\right) \cup Y_1^\alpha \supseteq \overline{T}_4 \cup \overline{T}_3 \cup Y_1^\alpha = \overline{T}_1 \cup Y_1^\alpha \supseteq \overline{T}_1,$$

i.e., the inclusions $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \supseteq \overline{T}_4$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_5^\alpha \cup Y_4^\alpha \cup Y_2^\alpha \supseteq \overline{T}_2$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_4^\alpha \cup Y_3^\alpha \cup Y_1^\alpha \supseteq \overline{T}_1$ are always hold. Further, it is to see, that the following conditions are true:

$$l\left(\ddot{Q}_{T_7}, T_7\right) = \cup\left(\ddot{Q}_{T_7} \setminus \{T_7\}\right) = \varnothing,\ \ T_7 \setminus l\left(\ddot{Q}_{T_7}, T_7\right) = T_7 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{Q}_{T_6}, T_6\right) = \cup\left(\ddot{Q}_{T_6} \setminus \{T_6\}\right) = \varnothing,\ \ T_6 \setminus l\left(\ddot{Q}_{T_6}, T_6\right) = T_6 \setminus \varnothing \neq \varnothing;$$
$$l\left(\ddot{Q}_{T_5}, T_5\right) = \cup\left(\ddot{Q}_{T_5} \setminus \{T_5\}\right) = T_7,\ \ T_5 \setminus l\left(\ddot{Q}_{T_5}, T_5\right) = T_5 \setminus T_7 \neq \varnothing;$$
$$l\left(\ddot{Q}_{T_3}, T_3\right) = \cup\left(\ddot{Q}_{T_3} \setminus \{T_3\}\right) = T_6,\ \ T_3 \setminus l\left(\ddot{Q}_{T_3}, T_3\right) = T_3 \setminus T_6 \neq \varnothing;$$
$$l\left(\ddot{Q}_{T_4}, T_4\right) = \cup\left(\ddot{Q}_{T_4} \setminus \{T_4\}\right) = T_4,\ \ T_4 \setminus l\left(\ddot{Q}_{T_4}, T_4\right) = T_4 \setminus T_4 = \varnothing;$$
$$l\left(\ddot{Q}_{T_2}, T_2\right) = \cup\left(\ddot{Q}_{T_2} \setminus \{T_2\}\right) = T_2,\ \ T_2 \setminus l\left(\ddot{Q}_{T_2}, T_2\right) = T_2 \setminus T_2 = \varnothing;$$
$$l\left(\ddot{Q}_{T_1}, T_1\right) = \cup\left(\ddot{Q}_{T_1} \setminus \{T_1\}\right) = T_1,\ \ T_1 \setminus l\left(\ddot{Q}_{T_1}, T_1\right) = T_1 \setminus T_1 = \varnothing,$$

i.e., T_7, T_6, T_5, T_3 are nonlimiting elements of the sets $\ddot{Q}(\alpha)_{T_7}$, $\ddot{Q}(\alpha)_{T_6}$, $\ddot{Q}(\alpha)_{T_5}$ and $\ddot{Q}(\alpha)_{T_3}$ respectively. By Statement c) of the Theorem 1.2 it follows, that the conditions $Y_7^\alpha \cap \overline{T}_7 \neq \varnothing$, $Y_6^\alpha \cap \overline{T}_6 \neq \varnothing$, $Y_5^\alpha \cap \overline{T}_5 \neq \varnothing$ and $Y_3^\alpha \cap \overline{T}_3 \neq \varnothing$ are hold. Since $Z_7 \subset Z_5$, $Z_6 \subset Z_3$ we have $Y_5^\alpha \cap \overline{T}_5 \neq \varnothing$ and $Y_3^\alpha \cap \overline{T}_3 \neq \varnothing$.

Therefore the following conditions are hold:

$$Y_7^\alpha \supseteq \overline{T}_7,\ Y_6^\alpha \supseteq \overline{T}_6,\ Y_7^\alpha \cup Y_5^\alpha \supseteq \overline{T}_5,\ Y_6^\alpha \cup Y_3^\alpha \supseteq \overline{T}_3,\ Y_5^\alpha \cap \overline{T}_5 \neq \varnothing,\ Y_3^\alpha \cap \overline{T}_3 \neq \varnothing$$

The lemma is proved.

Definition 2.1. Assume that $Q' \in \Sigma_2(X, 8)$. Denote by the symbol $R(Q')$ the set of all regular elements α of the semigroup $B_X(D)$, for which the semilattices Q' and Q are mutually α-isomorphic and $V(D, \alpha) = Q'$.

It is easy to see the number q of automorphism of the semilattice Q is equal to 2.

Theorem 2.2. Let $Q = \{T_7, T_6, T_5, T_4, T_3, T_2, T_1, T_0\} \in \Sigma_2(X, 8)$, $T_5 \cap T_3 = \varnothing$ and $\left|\Sigma_2(X, 8)\right| = m_0$. If X be finite set, and the XI-semilattice Q and $Q' = \{\overline{T}_7, \overline{T}_6, \overline{T}_5, \overline{T}_4, \overline{T}_3, \overline{T}_2, \overline{T}_1, \overline{T}_0\}$ are α-isomorphic, then

$$\left|R(Q')\right| = 2\cdot m_0\cdot\left(2^{\left|\overline{T}_5\setminus\overline{T}_1\right|}-1\right)\cdot\left(2^{\left|\overline{T}_3\setminus\overline{T}_2\right|}-1\right)\cdot 8^{\left|X\setminus\overline{T}_0\right|}$$

Proof. Assume that $\alpha\in R(Q')$. Then a quasinormal representation of a regular binary relation α has the form

$$\alpha=\left(Y_7^\alpha\times T_7\right)\cup\left(Y_6^\alpha\times T_6\right)\cup\left(Y_5^\alpha\times T_5\right)\cup\left(Y_4^\alpha\times T_4\right)\cup\left(Y_3^\alpha\times T_3\right)\cup\left(Y_2^\alpha\times T_2\right)\cup\left(Y_1^\alpha\times T_1\right)\cup\left(Y_0^\alpha\times T_0\right)$$

where $Y_7^\alpha,\ Y_6^\alpha,\ Y_5^\alpha,\ Y_3^\alpha\notin\{\varnothing\}$ and by Lemma 2.3 satisfies the conditions: X

$$Y_7^\alpha\supseteq\overline{T}_7,\ Y_6^\alpha\supseteq\overline{T}_6,\ Y_7^\alpha\cup Y_5^\alpha\supseteq\overline{T}_5,\ Y_6^\alpha\cup Y_3^\alpha\supseteq\overline{T}_3,\ Y_5^\alpha\cap\overline{T}_5\neq\varnothing,\ Y_3^\alpha\cap\overline{T}_3\neq\varnothing \tag{3}$$

Let f_α is a mapping the set X in the semilattice Q satisfying the conditions $f_\alpha(t)=t\alpha$ for all $t\in X$. $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$ are the restrictions of the mapping f_α on the sets $\overline{T}_7,\ \overline{T}_6,\ \overline{T}_3\setminus\overline{T}_2,\ \overline{T}_5\setminus\overline{T}_1,\ X\setminus\overline{T}_0$ respectively. It is clear, that the intersection disjoint elements of the set $\left\{\overline{T}_7,\overline{T}_6,\ \overline{T}_3\setminus\overline{T}_2,\ \overline{T}_5\setminus\overline{T}_1,\ X\setminus\overline{T}_0\right\}$ are empty set and $\overline{T}_7\cup\overline{T}_6\cup\left(\overline{T}_3\setminus\overline{T}_2\right)\cup\left(\overline{T}_5\setminus\overline{T}_1\right)\cup\left(X\setminus\overline{T}_0\right)=X$.

We are going to find properties of the maps $f_{1\alpha}$, $f_{2\alpha}$, $f_{3\alpha}$, $f_{4\alpha}$, $f_{5\alpha}$.

1) $t\in\overline{T}_7$. Then by Property (3) we have $t\in\overline{T}_7\subseteq Y_7^\alpha$, *i.e.*, $t\in Y_7^\alpha$ and $t\alpha=\overline{T}_7$ by definition of the set Y_7^α. Therefore $f_{1\alpha}(t)=T_7$ for all $t\in\overline{T}_7$.

2) $t\in\overline{T}_6$. Then by Property (3) we have $t\in\overline{T}_6\subseteq Y_6^\alpha$, *i.e.*, $t\in Y_6^\alpha$ and $t\alpha=\overline{T}_6$ by definition of the set Y_6^α. Therefore $f_{2\alpha}(t)=T_6$ for all $t\in\overline{T}_6$.

3) $t\in\overline{T}_3\setminus\overline{T}_2$. Then by Property (3) we have $t\in\overline{T}_3\setminus\overline{T}_2\subseteq\overline{T}_3\subseteq Y_6^\alpha\cup Y_3^\alpha$, *i.e.*, $t\in Y_6^\alpha\cup Y_3^\alpha$ and $t\alpha\in\{T_6,T_3\}$ by definition of the sets Y_6^α and Y_3^α. Therefore $f_{3\alpha}(t)\in\{T_6,T_3\}$ for all $t\in\overline{T}_3\setminus\overline{T}_2$.

Preposition we have that $Y_3^\alpha\cap\overline{T}_3\neq\varnothing$, *i.e.* $t_3\alpha=T_3$ for some $t_3\in\overline{T}_3$. If $t_3\in\overline{T}_2$, then $t_3\in Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_2^\alpha$. So $t_3\alpha=\{T_7,T_6,T_5,T_4,T_2\}$ by definition of the sets Y_7^α, Y_6^α, Y_5^α, Y_4^α, Y_2^α. The condition $t_3\alpha=\{T_7,T_6,T_5,T_4,T_2\}$ contradict of the equality $t_3\alpha=T_3$, while $T_3\notin\{T_7,T_6,T_5,T_4,T_2\}$. Therefore, $f_{3\alpha}(t_3)=T_3$ for some $t\in\overline{T}_3\setminus\overline{T}_2$.

4) $t\in\overline{T}_5\setminus\overline{T}_1$. Then by Property (3) we have $t\in\overline{T}_5\setminus\overline{T}_1\subseteq\overline{T}_5\subseteq Y_7^\alpha\cup Y_5^\alpha$, *i.e.*, $t\in Y_7^\alpha\cup Y_5^\alpha$ and $t\alpha\in\{T_7,T_5\}$ by definition of the sets Y_7^α and Y_5^α. Therefore $f_{4\alpha}(t)\in\{T_7,T_5\}$ for all $t\in\overline{T}_5\setminus\overline{T}_1$.

Preposition we have that $Y_5^\alpha\cap\overline{T}_5\neq\varnothing$, *i.e.* $t_4\alpha=T_5$ for some $t_4\in\overline{T}_5$. If $t_4\in\overline{T}_1$, then $t_4\in Y_7^\alpha\cup Y_6^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_1^\alpha$. So $t_4\alpha=\{T_7,T_6,T_4,T_3,T_1\}$ by definition of the sets Y_7^α, Y_6^α, Y_4^α, Y_3^α, Y_1^α. The condition $t_4\alpha=\{T_7,T_6,T_4,T_3,T_1\}$ contradict of the equality, $t_4\alpha=T_5$, while $T_5\notin\{T_7,T_6,T_4,T_3,T_1\}$. Therefore, $f_{4\alpha}(t_4)=T_5$ for some $t\in\overline{T}_5\setminus\overline{T}_1$.

5) $t\in X\setminus\overline{T}_0$. Then by definition quasinormal representation binary relation α and by Property (3) we have $t\in X\setminus\overline{T}_0\subseteq X=Y_7^\alpha\cup Y_6^\alpha\cup Y_5^\alpha\cup Y_4^\alpha\cup Y_3^\alpha\cup Y_2^\alpha\cup Y_1^\alpha\cup Y_0^\alpha$, *i.e.* $t\alpha\in\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}$ by definition of the sets Y_7^α, Y_6^α, Y_5^α, Y_4^α, Y_3^α, Y_2^α, Y_1^α, Y_0^α. Therefore $f_{5\alpha}(t)\in\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}$ for all $t\in X\setminus\overline{T}_0$.

Therefore for every binary relation $\alpha\in R(Q')$ exist ordered system $(f_{1\alpha},f_{2\alpha},f_{3\alpha},f_{4\alpha},f_{5\alpha})$. It is obvious that for different binary relations exist different ordered systems.

Let $f_1:\overline{T}_7\to\{T_7\}$, $f_2:\overline{T}_6\to\{T_6\}$, $f_3:\overline{T}_3\setminus\overline{T}_2\to\{T_6,T_3\}$, $f_4:\overline{T}_5\setminus\overline{T}_1\to\{T_7,T_5\}$, $f_5:X\setminus\overline{T}_0\to Q$ are such mappings, which satisfying the conditions:

6) $f_1(t)\in\{T_7\}$ for all $t\in\overline{T}_7$;

7) $f_2(t)\in\{T_6\}$ for all $t\in\overline{T}_6$;

8) $f_3(t)\in\{T_6,T_3\}$ for all $t\in\overline{T}_3\setminus\overline{T}_2$ and $f_3(t_3)=T_3$ for some $t_3\in\overline{T}_3\setminus\overline{T}_2$;

9) $f_4(t)\in\{T_7,T_5\}$ for all $t\in\overline{T}_5\setminus\overline{T}_1$ and $f_4(t_4)=Z_4$ for some $t_4\in\overline{T}_5\setminus\overline{T}_1$;

10) $f_5(t)\in\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}$ for all $t\in X\setminus\overline{T}_0$.

Now we define a map f of a set X in the semilattice Q, which satisfies the following condition:

$$f(t)=\begin{cases} f_1(t), & \text{if } t\in\overline{T}_7,\\ f_2(t), & \text{if } t\in\overline{T}_6,\\ f_3(t), & \text{if } t\in\overline{T}_3\setminus\overline{T}_2,\\ f_4(t), & \text{if } t\in\overline{T}_5\setminus\overline{T}_1,\\ f_5(t), & \text{if } t\in X\setminus\overline{T}_0.\end{cases}$$

Now let $\beta=\bigcup_{x\in X}(\{x\}\times f(x))$, $Y_i^\beta=\{t\,|\,t\beta=T_i\}$ $(i=1,2,\cdots,5)$. Then binary relation β is written in the form

$$\beta=(Y_7^\beta\times T_7)\cup(Y_6^\beta\times T_6)\cup(Y_5^\beta\times T_5)\cup(Y_4^\beta\times T_4)\cup(Y_3^\beta\times T_3)\cup(Y_2^\beta\times T_2)\cup(Y_1^\beta\times T_1)\cup(Y_0^\beta\times T_0)$$

and satisfying the conditions:

$$Y_7^\beta\supseteq\overline{T}_7,\ \ Y_6^\beta\supseteq\overline{T}_6,\ \ Y_7^\beta\cup Y_5^\beta\supseteq\overline{T}_5,\ \ Y_6^\beta\cup Y_3^\beta\supseteq\overline{T}_3,\ \ Y_5^\beta\cap\overline{T}_5\neq\varnothing,\ \ Y_3^\beta\cap\overline{T}_3\neq\varnothing$$

From this and by Lemma 2.3 we have that $\beta\in R(Q')$.

Therefore for every binary relation $\alpha\in R(Q')$ and ordered system $(f_{1\alpha},f_{2\alpha},f_{3\alpha},f_{4\alpha},f_{5\alpha})$ exist one to one mapping.

By Theorem 1.1 the number of the mappings $f_{1\alpha},f_{2\alpha},f_{3\alpha},f_{4\alpha},f_{5\alpha}$ are respectively:

$$1,\ 1,\ 2^{|\overline{T}_3\setminus\overline{T}_2|}-1,\ 2^{|\overline{T}_5\setminus\overline{T}_1|}-1,\ 8^{|X\setminus\overline{T}_0|}$$

(see ([1], Corollary 1.18.1), ([2], Corollary 1.18.1)).

The number of ordered system $(f_{1\alpha},f_{2\alpha},f_{3\alpha},f_{4\alpha},f_{5\alpha})$ or number regular elements can be calculated by the formula

$$|R(Q')|=2\cdot m_0\cdot\left(2^{|\overline{T}_5\setminus\overline{T}_1|}-1\right)\cdot\left(2^{|\overline{T}_3\setminus\overline{T}_2|}-1\right)\cdot 8^{|X\setminus\overline{T}_0|}$$

(see ([1], Theorem 6.3.5), ([2], Theorem 6.3.5)).

The theorem is proved.

Corollary 2.1. Let $Q=\{T_7,T_6,T_5,T_4,T_3,T_2,T_1,T_0\}\in\Sigma_2(X,8)$, $T_5\cap T_3=\varnothing$. If X be a finite set and $E_X^{(r)}(Q)$ be the set of all right units of the semigroup $B_X(Q)$, then the following formula is true

$$\left|E_X^{(r)}(Q)\right|=\left(2^{|T_5\setminus T_1|}-1\right)\cdot\left(2^{|T_3\setminus T_2|}-1\right)\cdot 8^{|X\setminus T_0|}$$

Proof: This corollary immediately follows from Theorem 2.2 and from the ([1], Theorem 6.3.7) or ([2], Theorem 6.3.7).

The corollary is proved.

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