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The Kakeya Problem

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Abstract

This research paper concentrates on the Kakeya problem. After the introduction of historical issue, we provide a thorough presentation of the results of Kakeya problem with some examples of the early solutions as well as the proof of the final outcome of this problem, the solution of which is known as Besicovitch Set. We give 3 different construction of Besicovitch set as well as the intuition of construction, which is related to iterated integral of 2-variable real function. We also give the Cunningham construction in which the area of a simply connected Kakeya set can also tend to 0. Furthermore, we generalize the process of generating a Kakeya set into a Kakeya dynamic. The definition of multiplicity enables us to estimate the area of a Kakeya set. In following discussion we provided a conjecture related to the solution in particular range. Finally, the derivation of the Kakeya problem is presented.

Keywords

Kakeya Needle Problem, Besicovitch Set

1. Introduction

1.1. History of Kakeya Problem

In 1917, Sōichi Kakeya asked a question: what is the smallest area which enables a unit line segment to rotate 180 degrees and return to the initial position in reversed direction? In honor of Kakeya, a compact set $E \subset \mathbb{R}^n$ in which the unit line segments can be found in every direction is defined to be *Kakeya set*.

$$\forall \xi \in S_{n-1}, \exists x \in \mathbb{R}^n, s.t.\ x + t\xi \in E, \forall t \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

where S_{n-1} is the notation for unit sphere in $\mathbb{R}^n$. The optimal solution for the Kakeya problem when $n = 2$ is constructed by Besicovich, known as *Besicovich set*.

1.2. History of Besicovitch

Abram Samotlvitch Besicovich made great contributions to the Kakeya problem. Described in [1], his family had seven children and they were living in a frugal life. The older ones earned money to support the younger ones. They all studied at the University of St. Petersburg and received high education. After his graduation, Besicovitch published his first paper about probability theory. Later in 1917, he became a professor in the University, which was destroyed in 1919 during the Civil War. However, Besicovitch locked books in the cellar and preserved most of the property, which later contributed to the re-establishment of the university after the liberation of Perm. In 1920, he returned to Leningrand and gave lectures on Pedagogical Institute for four years. However, from 1920, as the Russian revolution was launched, he was forced to lecture to workers who had weak mathematical backgrounds. To leave Russia, he decided to apply for Rockefeller Fellowship, a fellowship that enabled people to work abroad, but the offer was not obtained until 1924.

Accompanied with another mathematician, J.D. Tamarkin, Abram Besicovitch went to Copenhagen. In Copenhagen, he worked with Harald Bohr for a year. Then, he made his way for Oxford and stayed for several months with G.H. Hardy, who recognized his great analytical talent and enabled his lectureship at the University of Liverpool for 1926-1927. In 1927, he moved to Cambridge and became a college lecturer as well as a Fellow of Trinity.

Besicovitch passed most of his life in Britain. After retiring from the Rouse Ball Chair in 1958, he remained active in the field of mathematics as a visiting professor in the United States. After all, he died in his eightieth year on 2 November, 1970. Besicovitch is a successful mathematician who received several high standard awards and medals and his mathematical work is still valuable today and still influences the modern mathematical field.

1.3. The Structure of This Paper

This paper about the Kakeya problem makes an entire summary of the historical solutions of Kakeya problem. In Section 2, some basic examples of Kakeya sets are given and we shall see the basic techniques of constructing a Kakeya set with small area is by overlapping some basic Kakeya set. Intuitively, the more area they overlap, the smaller their total area would be. In Section 3, we illustrate the construction of a Besicovitch set in Besicovitch [2], a kind of Kakeya set which can achieve arbitrary small area, by introducing the Pal-join technique that enables the parallel transformation. After the construction of that set, we also discuss the connection of the Besicovitch set and the iterated integral. In Section 4, we summarize the simply connected Kakeya set which was discovered in

Cunningham [3] and again, find the basic ideas behind constructing them are still overlapping them as heavily as possible. In Section 5, we give a measure of the overlap in terms of the multiplicity of a Kakeya set. With the system constructed in Section 5, we are able to explain the reason that the Besicovitch set can achieve an arbitrary small area. After that, as a conjecture, we give some constraint under which a Kakeya set might not able to achieve a arbitrary small area and as a consequence, the deltoid become the optimal solution of the Kakeya problem. In Section 6, we provide the recent result and algebraic analog of the Kakeya problem under finite field context. One of the most famous results is the size of finite field Kakeya conjecture which has been proved in Divr [4]. Also, some other derivatives discussed in Pugh [5] and Furtner [6] are also included. The summary of this paper is in Section 7.

2. Some Examples of Kakeya Sets

2.1. Circle

Details. Rotate a unit needle centered at its midpoint for 180˚, forming a circle (see **Figure 1**).

Area calculation. Let the length of the needle be 1. The area of the circle is $S_i = \left(\frac{1}{2}\right)^2 \pi = \frac{\pi}{4} = 0.785$.

2.2. Curved Edge Triangle

Details. Combine three identical 60˚ sectors together to form a convex triangle. The edges of the sectors form an equilateral triangle. The needle starts from an edge of the triangle, then rotates 60˚ for three times in order to reverse itself (see **Figure 2**).

Area calculation. Let the length of the needle be 1, so the length of the side of each sector 1. The area of each sector is $S_i = \frac{\pi}{6}$, where i = 1, 2, 3. The area of inner triangle is $S' = \frac{\sqrt{3}}{4}$. The total area of the convex triangle is

$$S = 2\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + \frac{\pi}{6} = 0.705.$$

2.3. Equilateral Triangle

Details. Combine three identical 60˚ sectors together to form an equilateral triangle. The needle starts from one side of the triangle with one end of needle at a vertex of the triangle, rotates to another side of the triangle, then slides a little to the another vertex of the triangle. Repeat the process three times. Then, the needle reverses itself (see **Figure 3**).

Area calculation. Assume the length of the needle be 1, so the height of the equilateral triangle equals to 1. The area of the triangle is $S = \frac{1}{\sqrt{3}} = 0.577$.

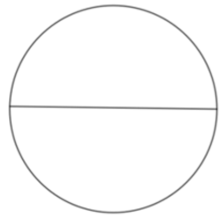


Figure 1.

Figure 2.

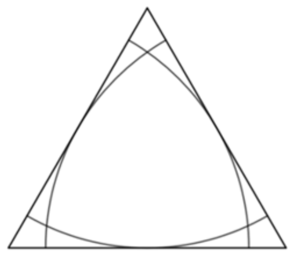


Figure 3.

2.4. Sunshape

Details. Since both the equilateral triangle and the convex triangle are both composed of three identical sectors with 60° (see Figure 4), which can construct a semicircle without overlapping. Then, we can cut the semicircle into smaller sectors and construct them to a smaller figure. For example, the figure composed of seven sectors is similar to the seven-point star except the seven small sectors between the outer triangles (see Figure 5).

Area calculation. Since the figure was composed of the a n-point star and n sectors, divide the area into two parts for calculation. Let the angle of half of the outer triangles be θ, then $\theta = \frac{2\pi}{2n+1}$. Then make an angle bisector from one of the vertex A to intersect the n-point star with point B, and intersect the arc of the sector with point C. Let the length of AB be l (see Figure 6). Since the calculation of small sector is complicated, make a line tangent to the arc of the sector at point C to form a small triangle. Let the length of the needle be 1, then $AC = 1$, $BC = 1 - l$. The area S of the whole figure is

$$S = n \cdot l^2 \cdot \frac{\sin^2(\theta)}{\sin(2\theta)} \cdot \left(1 - \frac{\sin^2(\theta)}{\sin^2(2\theta)}\right) + n \cdot (1-l)^2 \cdot \tan(3\theta)$$

Figure 4.

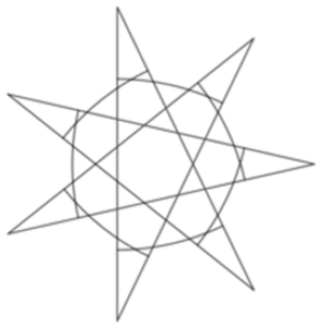


Figure 5.

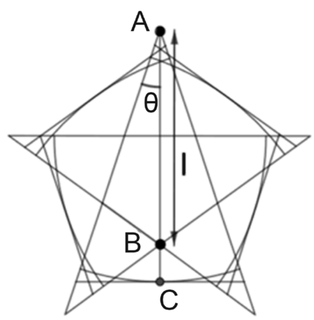


Figure 6.

Then the area of figure with five sectors is

$$S(l)=\frac{\sqrt{2}\left(3\left(3\sqrt{5}+5\right)l^2-20\left(\sqrt{5}+1\right)l+10\left(\sqrt{5}+1\right)\right)}{4\sqrt{5-\sqrt{5}}}$$

Then the minimum area of the figure with five sectors is 0.542 when $l = 0.921$.

When the semicircle is cut into n pieces, it can achieve its minimum area. When n goes to ∞, θ goes to 0. The area of the shape is

$$S=\frac{3}{16}\pi l^2+\frac{3}{2}\pi\left(1-l\right)^2$$

The minimum area of this figure is 0.524 when $l=\frac{8}{9}$.

2.5. Deltoid

Details. Historically, deltoid was considered to be the optimal solution of the

Kakeya problem. One can intuitively understand the formation of a deltoid in [7]. It's also easy to see how it is qualified for a Kakeya set: the needle starts from the middle of the deltoid which one of the ends is at a vertex of the deltoid and another end is at the middle point of the side. The needle rotates along and keeps tangent to the side. The same process repeats three times. The needle reverses itself (see Figure 7).

Area calculation. Build an *x*-axis and a *y*-axis. The function of deltoid curve can be presented on the coordinate which is

$$x(t) = (R-r)\cos t + r\cos\left(\frac{(R-r)t}{r}\right)$$

$$y(t) = (R-r)\sin t - r\sin\left(\frac{(R-r)t}{r}\right)$$

where R represents the radius of the large circle which is $\frac{3}{2}$, and r represents the radius of the rolling circle which is $\frac{1}{2}$. Take the data into the function and then we can calculate the area *S*:

$$S = \int y\mathrm{d}x = \int_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} \frac{1}{8}\left(\sin^2(2t) + 2\sin^2 t\cos t - 4\sin t\cos^2 t - 2\sin t\cos t\right)$$

$$= \frac{1}{8}\left(\frac{1}{2}t - \frac{1}{8}\sin 4t + \frac{2}{3}\sin^3 t + \frac{4}{3}\cos^3 t + \frac{1}{2}\cos 2t\right)\Bigg|_{-\frac{3\pi}{2}}^{\frac{3\pi}{2}} = \frac{\pi}{8}$$

3. Besicovitch Set

3.1. Translation between Parallel Line

To deal with the Kakeya problem, the trick named *Pàl joins* is established to achieve the minimum rotating area.

Given two parallel lines l_1, l_2, Let x_1 be any point in l_1, and x_2 be a point in l_2. Connect the two points and name the line across x_1, x_2 as l_{ab}. Let the angle between l_{ab} and l_1 (or l_2) be θ. Assume that the original position of the unit segment is in l_1. Move that unit segment such that one of its endpoint reaches x_1. Then rotate the segment to coincide with l_{ab}, and slide to x_2. Finally, rotate the unit segment to l_2. Since the area cost depends on θ, when the theta becomes extremely small, the unit line segment can "jump" from l_1 to l_2 cost a very small area (see Figure 8). The following lemma is the proof of this strategy.

1 Lemma. Let l_1, l_2 be parallel lines. $\forall \epsilon > 0$, $\exists$ a compact set E s.t. $|E| < \epsilon$, in which any unit line segment can be moved continuously from l_1 to l_2.

Proof. Let d be the distance between l_1 and l_2 and d' be the distance between the projection of x_1 onto l_2 and x_2. The angle which between l_{ab} and l_1, l_2 is θ. Thus

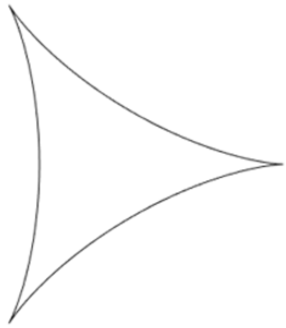


Figure 7.

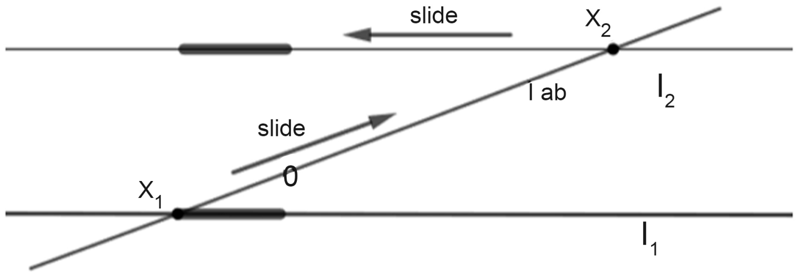


Figure 8.

$$d' = \frac{d}{\tan\theta}$$

$\forall \epsilon > 0$, because θ can be arbitrarily small,

$$d' > \frac{d}{\tan \pi\epsilon}$$

Thus

$$\theta = \tan^{-1}\frac{d}{d'} < \pi\epsilon$$

The congruent sectors between l_{ab} and l_1, l_2 are the swept areas that are not negligible. Let M represents the area that the segment travels as it moves along the straight lines. Let S_1, S_2 be the sectors between l_{ab} and l_1, l_2 with unit radius. Take $E = M \cup S_1 \cup S_2$. M can be arbitrary small and S_1, $S_2 < \frac{\epsilon}{2}$, so $E < \epsilon$. □

3.2. Besicovitch Sets

The lemma above implies the possibility to construct a compact set which is Lebesgue measure zero and contains unit line segments in every direction. There are two methods to construct the Besicovitch set which satisfies the conditions.

Method 1. First, we show the original version Besicovitch set which has been simplified to be the *Perron tree*.

2 Lemma. Given a triangle T with height h and bottom length $2b$ (see **Figure 9**), divide it through the midline into two triangles T_1, T_2 with same bottom length b (see Figure 10). Slide T_1, T_2 along the bottom line at a contrary direction such that the overlapped length of their bottom is $2(1-\alpha)b$, where $\frac{1}{2} \le \alpha \le 1$. Let the area of T be $|T|$. The area S of this new figure, containing two small triangles A_1, A_2 above and a triangle T_1, which is similar to T, at the base (see Figure 11) is

$$S = \left(\alpha^2 + 2(1-\alpha)^2\right)|T|$$

Proof. The bottom side length of T_1 is $2\alpha b$. So

$$|T_1| = \alpha^2 |T|$$

The line, parallel to the bottom side and across the vertex of T_1, divides A_1, A_2 into four triangles $A_1 1$, $A_1 2$, $A_2 1$, $A_2 2$ (see Figure10). $A_1 1$ is congruent to $A_2 2$, and $A_1 2$ is congruent to $A_2 1$. The bottom length of them is $(1-\alpha)b$ and the height is $(1-\alpha)h$. Thus the area of A_1 and A_2 is

$$|A_1| = |A_2| = (1-\alpha)|T|$$

Thus the total area of T_1 is

$$S = |A_1| + |A_2| + |T_1| = \left(\alpha^2 + 2(1-\alpha)^2\right)|T| \qquad \square$$

Based on this construction technique and the lemma above, we can construct the Perron tree as follows:

Given a triangle with height 1 (see Figure 12), divide the bottom of 2^n equal pieces and get 2^n triangles (see Figure 13).

Step 1, move the adjacent triangles T_{2i-1}, T_{2i} ($1 \le i \le 2^{n-1}$) with the same technique and get 2^{n-1} figures called S_i^1 (see Figure 14 & Figure 15). The top two small triangles are called A_{2i-1}^1, A_{2i}^1 and basal triangle is called T_i^1 (see Figure 16). One side of T_{2i-1} is parallel and equal to the other side of T_{2i}, so S_i^1 is translated such that T_i^1 forms a triangle called T^1 which is similar to T.

Step 2, move the adjacent figures S_{2i-1}^1, S_{2i}^1 ($1 \le i \le 2^{n-2}$) to form S_i^2 and name the basal triangle as T_i^2 (see Figure 17 & Figure 18). Translate S_i^1 to let T_i^1 form a triangle which is similar to T called T^2.

⋮

Step r ($2 \le r \le n$), move the adjacent figures S_{2i-1}^{r-1}, S_{2i}^{r-1} ($1 \le i \le 2^{n-r}$) to form S_i^r. and name the basal triangle as T_i^r. Translate S_i^r to let T_i^r form a triangle named T^r which is similar to T (see Figure 19).

⋮

In the final step, we can obtain a single figure with 2^n small triangles above and one basal triangle T^n. This is the *Perron tree* (see Figure 20).

3 Theorem. The measure of the *Perron tree* can be arbitrary small.

Proof. In the first step, from lemma() we have

$$S_i^1 = \left(\alpha^2 + 2(1-\alpha)^2\right)|T_i^1|$$

From the translation of S_i^1 we have

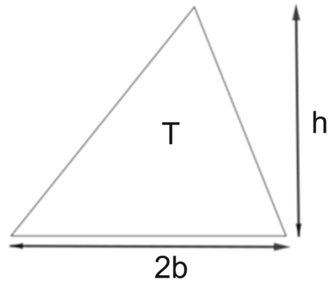


Figure 9.

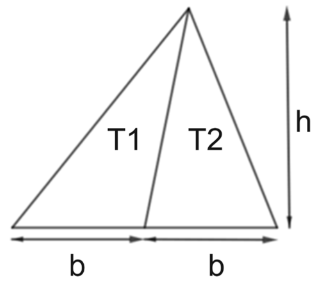


Figure 10.

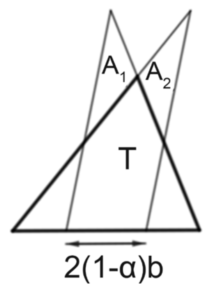


Figure 11.

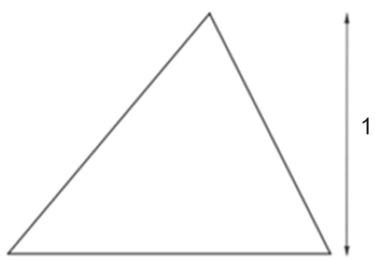


Figure 12.

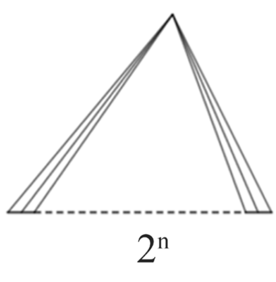


Figure 13.

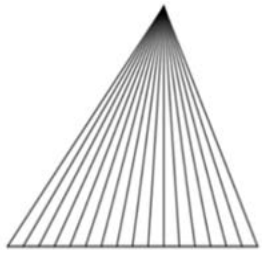


Figure 14.

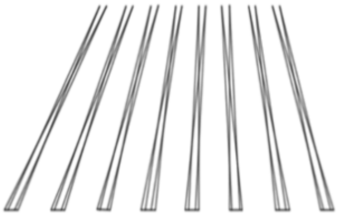


Figure 15.

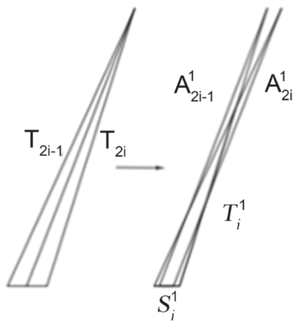


Figure 16.

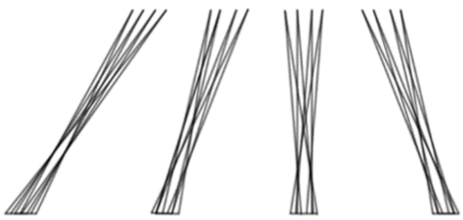


Figure 17.

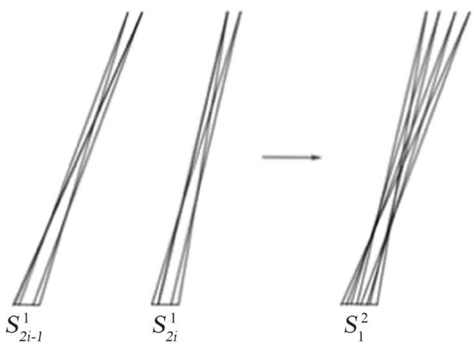


Figure 18.

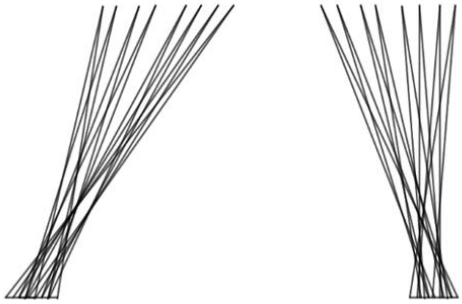


Figure 19.

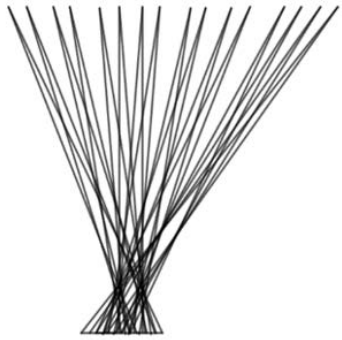


Figure 20.

$$\left|T^1\right| = \alpha^2 \left|T\right|$$

Because there are some overlapped parts,

$$\sum_{i=1}^{2^{n-1}} S_i^1 \le \left(\alpha^2 + 2(1-\alpha)^2\right)\left|T\right|$$

In the second step, we have

$$\left|T^2\right| = \alpha^4 \left|T\right|$$

$$\sum_{i=1}^{2^{n-2}} S_i^2 \le \left(\alpha^4 + 2(1-\alpha)^2 + 2(1-\alpha)^2 \alpha^2\right)\left|T\right|$$

$\vdots$

In the r-th step ($2 \le r \le n$), we have

$$\sum_{i=1}^{2^{n-r}} S_i^r \le \left(\alpha^{2r} + 2(1-\alpha)^2 \sum_{i=1}^{r-1} \alpha^{2i}\right)\left|T\right|$$

$\vdots$

In the final step, for the area S of the single figure,

$$\begin{aligned} S &\le \left(\alpha^{2n} + 2(1-\alpha)^2 \sum_{i=1}^{n-1} \alpha^{2i}\right)\left|T\right| \le \left(\alpha^{2n} + 2(1-\alpha)^2 \sum_{i=1}^{\infty} \alpha^{2i}\right)\left|T\right| \\ &= \left(\alpha^{2n} + \frac{2(1-\alpha)^2}{1-\alpha^2}\right)\left|T\right| = \left(\alpha^{2n} + \frac{2(1-\alpha)}{1+\alpha}\right)\left|T\right| \\ &\le \left(\alpha^{2n} + 2(1-\alpha)\right)\left|T\right| = M \end{aligned}$$

when $\alpha \to 1$ and $nT \to \infty$, $\forall \epsilon > 0$, $M < \epsilon$. Thus S can be arbitrary small. □

Method 2. The other attempt of constructing such set is simpler in a tricky way.

Given a triangle T with vertex θ and bottom length b (see **Figure 21**), cut the triangle from its angle bisector into two pieces marked as A_{11}, B_{11} (see **Figure 22**).

Move the two pieces along the bottom line at the opposite direction such that the overlapped length at the bottom is $(1-\alpha)b$, where $0 \le \alpha \le 1$. Let the new figure be set C_1. The basal triangle similar to T is named as T'. Thus the bottom length of T' is αb. Let the vertex of triangle A_{11} be a_{11}, and the vertex of triangle B_{11} be b_{11}. The angles of those two vertex are both $\frac{\theta}{2}$ (see **Figure 23**).

Extend the sides of A_{11} to points a_{21}, a_{22} such that the distances between a_{21}, a_{22} and a_{11} are 1. Let two lines pass the point a_{21} and a_{22} and be parallel to the angle bisector of A_{11} then we can obtain two small triangles named A_{21}, A_{22}. Repeat the steps above for B_{11} and get other two triangles B_{21}, B_{22}. We call the small triangle "horn". The apex angles of those four triangles are $\frac{\theta}{4}$. Set $C_2 = \{A_{21}, A_{22}, B_{21}, B_{22}\}$ (see **Figure 24**).

Extend the side of A_{21}, A_{22}, B_{21}, B_{22} and get eight smaller "horns", named A_{31}, A_{32}, A_{33}, A_{34}, B_{31}, B_{32}, B_{33}, B_{34}. The set contains them called C_3 (see **Figure 25**).

Continually repeat the manipulations above, we can get a set E like a "tree". $\forall n \in \mathbb{N}$, C_n contains 2^n "horns", whose apex corners are $\frac{\theta}{2^n}$.

4 Lemma. $\forall n \in \mathbb{N}$, $n \ne 1$, $\forall A_{ni}, A_{nj}, B_{nr}, B_{ns} \in C_n$, where $i, j, r, s \in [1, 2^{n-1}]$ and $i \ne j$, $r \ne s$, $A_{ni} \cong A_{nj} \cong B_{nr} \cong B_{ns}$.

Proof. For each triangle in C_n, because its bottom is parallel to the angle bisector of the corresponding triangle in C_{n-1}, it must be an isosceles triangle with basal angle $\frac{\theta}{2^n}$ and the lengths of the isosceles sides are 1 (see **Figure 26**). Also, because the third angle of them are $\pi - \frac{\theta}{2^{n-1}}$, every triangle in C_n satisfies the condition of the congruent theorem "ASA". Thus, every triangle in C_n is congruent. □

5 Lemma. $\forall n \in \mathbb{N}$, $n \ne 1$, assume the area of C_n is c_n, then $c_n < 2\theta$.

Proof. For each "horn" in C_n, the area of it is less than the area sum of two sectors with radius 1 and angle $\frac{\theta}{2^n}$. Therefore, $c_n < 2^n \left(2\frac{\theta}{2^n}\right) = 2\theta$. □

Assume that the area of the whole overlapped graph of A_{11}, B_{11} is c_1, which is a constant. So, for the total area of the "tree",

$$c = c_1 + \sum_{i=2}^{n} c_n < c_1 + \sum_{i=2}^{n} 2\theta = c_1 + 2(n-1)\theta. \quad \square$$

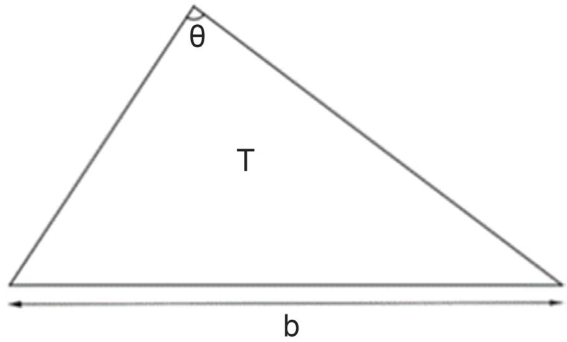


Figure 21.

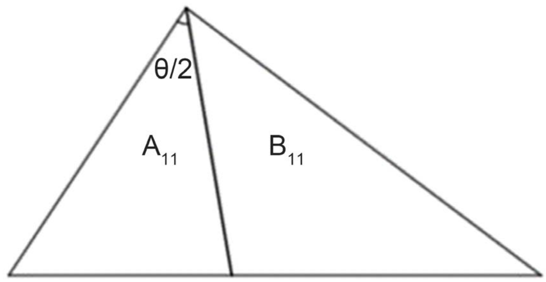


Figure 22.

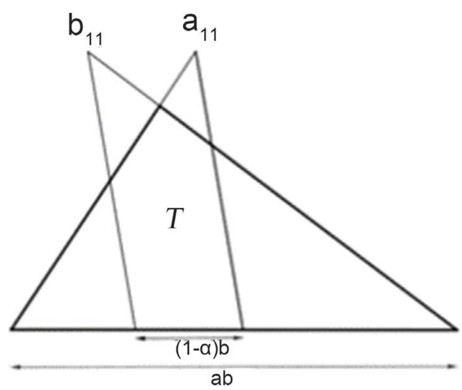
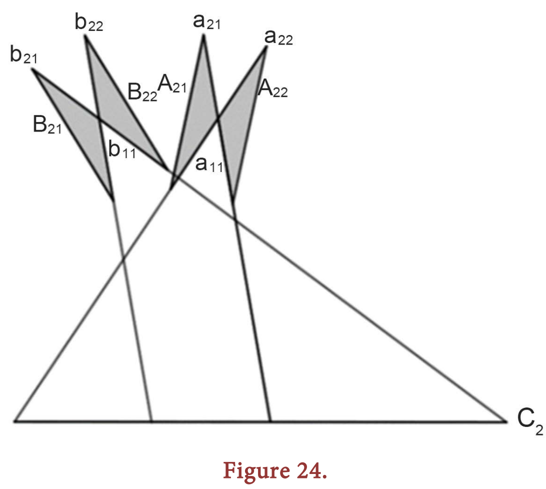


Figure 23.



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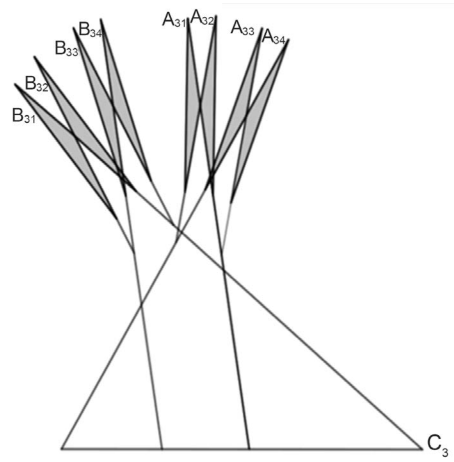

Figure 25.

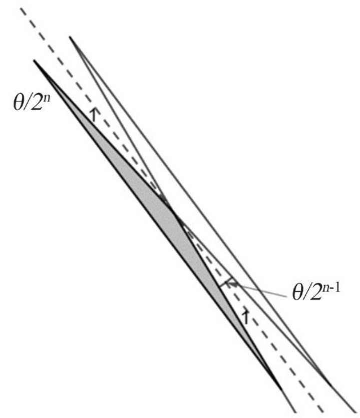

Figure 26.

Let $h=\min\left\{d\left(a_{ni},x_{ni}\right),d\left(b_{nj},y_{nj}\right)\right\}$ ($1\le i,j\le 2^{n-1}$), where a_{ni}, b_{ni} are the vertexes of the triangles in C_n satisfies that the segment which connected itself and its projection point is in the "tree", and x_{ni}, y_{nj} are the projection points. For the chosen vertex *P*, the extension cord of its adjacent side which is length 1 insects with the bottom at Q such that $d(P,Q)=(n-1)+a$, where $a>0$. Let the angle between the extension cord and the bottom be α. Thus, $h=\left[(n-1)+a\right]\sin\alpha$ (see Figure 27).

Compress the whole "tree" proportionally to obtain a new set E' which is similar to the previous set E and h becomes h', where $h'=1$. Let the basal triangle be T''. After the compression, for the whole area c',

$$\frac{c'}{c}=\frac{h'^2}{h^2}=\frac{1}{\left[(n-1+a)\sin\alpha\right]^2}$$

$$c'=\frac{c}{\left[(a+n-1)\sin\alpha\right]^2}\le\frac{c_1+2(n-1)\theta}{\left[(a+n-1)\sin\alpha\right]^2}=M$$

$$\lim_{n\to\infty}M=\lim_{n\to\infty}\frac{c_1+2(n-1)\theta}{\left[(a+n-1)\sin\alpha\right]^2}=\lim_{n\to\infty}\frac{2}{2\sin\alpha^2(a+n-1)}=0$$

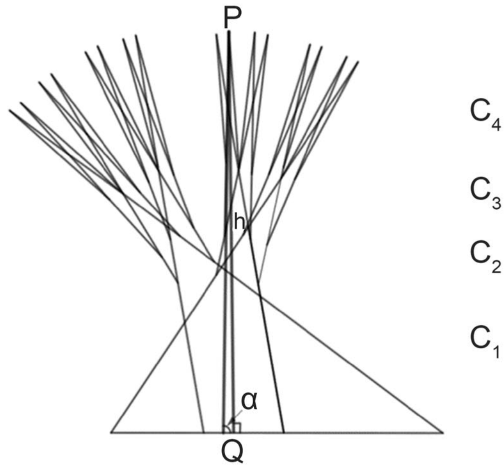

Figure 27.

Therefore, E' achieves the arbitrary small area.

6 Lemma. E' contains a unit line segment in every direction in θ.

Proof. Assume that after compressing, C_n becomes C_n'. Because E' is similar to *E*, C_n' is similar to *C*. For set E', $\forall$ set C_n', the sum of the apex is θ. Fix one endpoint of the unit line segment at a_{ni} or b_{nj}, where $1 \le i, j \le 2^{n-1}$. Assume the angle between the unit segment and the left side of T'' be β (see Figure 28).

First, we prove that every $\beta \in [0,\theta]$ can be found in E'. Note that every adjacent triangle in C_n has a parallel and equal side. Let I_{bj} = {β: β can be taken when unit segment rotates inside B_{nj}} and I_{ai} = {β: β can be taken when unit segment rotates inside A_{ni}}. Then we have

$$I_{bj} = \left[\theta - \frac{j\theta}{2^n}, \theta - \frac{(j-1)\theta}{2^n}\right], I_{b1} = \left[\theta - \frac{\theta}{2^n}, \theta\right], I_{bn} = \left[0, \theta - \frac{(n-1)\theta}{2^n}\right]$$

$$I_{ai} = \left[\frac{\theta}{2} - \frac{i\theta}{2^n}, \frac{\theta}{2} - \frac{(i-1)\theta}{2^n}\right], I_{a1} = \left[\frac{\theta}{2} - \frac{\theta}{2^n}, \theta\right], I_{an} = \left[0, \frac{\theta}{2} - \frac{(n-1)\theta}{2^n}\right]$$

Therefore,

$$[0,\theta] \subset (\cup I_{ai}) \cup (\cup I_{bj})$$

Then, we prove that E' contain a unit segment every direction in θ.

Because h is the minimum distance from the apex to the bottom, the unit segment can reach the bottom only when the endpoint is P' (corresponding to *P*). Thus the unit segment can be contained in every direction in θ in set E'. □

This lemma indicates that we can rotate the unit segment for every value in θ with the method of *Pàl joins*. The following is the specific manipulation.

Without the loss of generality, let the unit segment star at the left side. First use the parallel lemma to "jump" to the 2-nd triangle, rotate $\frac{\theta}{2^n}$, and then "jump" to 3-rd triangle, continually operate and then the unit segment can arrive the right side (see Figure 29).

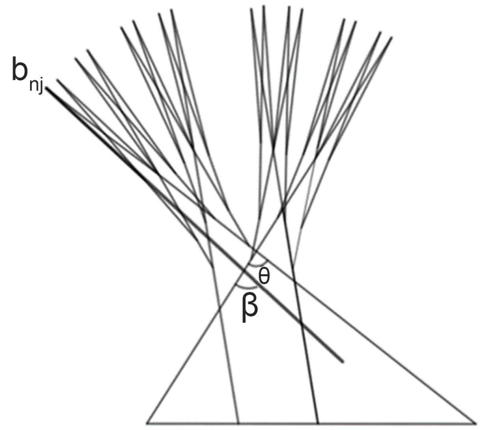

Figure 28.

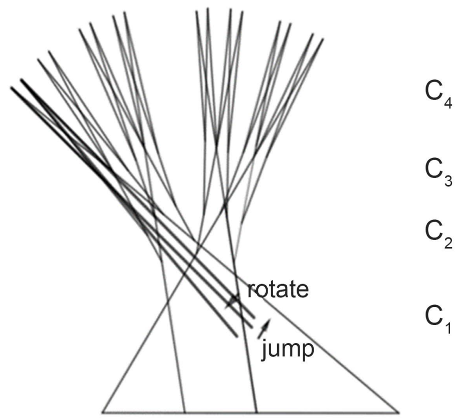

Figure 29.

The difference between Kakeya set and Besicovitch set is that Kakeya set permits movement. The detailed definition of movement will be discussed in Section 5.

Method 3. Instead of cutting triangle to form a Besicovitch sets, cutting sector will reduce the waste of cutting triangle (see **Figure 30**).

Inscribe a sector with the radius the same as the height of the isosceles triangle. Since when rotating the needle in the Besicovitch sets, the needle actually only uses the area of sectors without using the whole area of the triangle. So, there are wastes of the areas in using the triangle Besicovitch sets.

Assume a big triangle with the height of 1 is cut into 2^n pieces, and T is the set of all small triangles. Then, inscribe a sector with the radius the same as the height of the triangle. Then, cut the sectors into 2^n pieces, such that each small piece of the sector is contained in each small triangle. Assume S is the set of all small sectors. So, $S \subset T$.

Construct a triangle Besicovitch set, and then take out all the area of $T \setminus S$ to form a sectorial Besicovitch set. Since the areas of the triangle Besicovitch set equals to ϵ and $S \subset T$, the area of the sectorial Besicovitch set less than ϵ (see **Figures 31-33**).

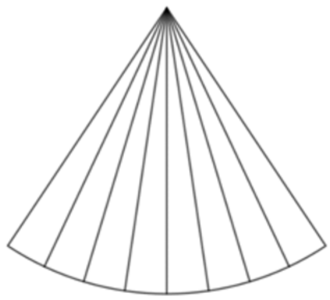

Figure 30.

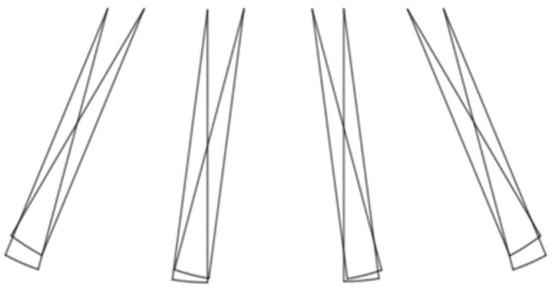

Figure 31.

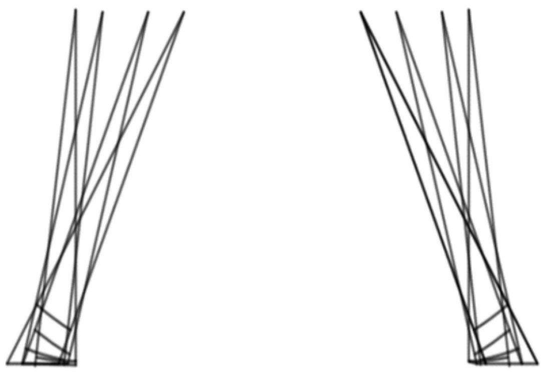

Figure 32.

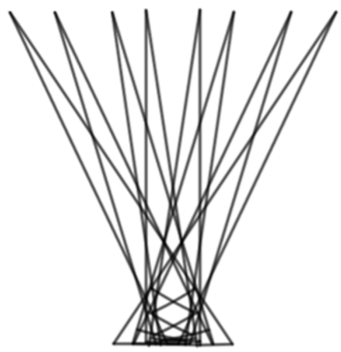

Figure 33.

3.3. The Origin of Besicovitch Set

Besicovitch noticed that for a Riemann integrable function f defined on the real plane $\mathbb{R}^2$, the existence of $\iint f(x,y)\mathrm{d}x\mathrm{d}y$ does not always imply the existence of $\int\left[\int f(x,y)\mathrm{d}x\right]\mathrm{d}y$. For example, let $f(x,y)=1$ if $x\in\mathbb{Q}$, $y=0$, and $f(x,y)=0$ otherwise. This function is zero except on a single line. Therefore, the discontinuity points comprise a planar zero set, and thus it is Riemann integrable on the plane. However, it is not Riemann integrable on the slice.

For the case above, a simple manipulation of rotating the coordinate would transform the function into a Riemann integrable function that could be integrated by iterated integrals. Besicovitch wondered if there exists a Riemann integrable function *f* defined on the plane which is free from the choice of orthogonal coordinate axes, such that the iterated integral $\int\left[\int f(x,y)\mathrm{d}x\right]\mathrm{d}y$ cannot substitute the Riemann integral $\iint f(x,y)\mathrm{d}x\mathrm{d}y$ for all possible linear coordinate systems.

He found a counterexample by constructing a compact zero set that contains a unit line segment in every direction, known as the Besicovitch set. The characteristic function χ_B is Riemann integrable since the set of discontinuity points is a zero set in plane.

For example, let B = Besicovitch set and let $A=(\mathbb{Q}\times\mathbb{R})\bigcup(\mathbb{R}\times\mathbb{Q})$, $f=\chi_{A\cap B}$. We can translate B in the *y*-direction so that some horizontal segments $\sigma_0\in B$ have rational *y*-coordinates y_0. Thus $\chi_{\mathbb{R}\times\mathbb{Q}}=1$ on σ_0 and $f(x,y_0)=\chi_{\mathbb{Q}}(x)$, which is not Riemann integrable as a function of *x*. Since $A\bigcap B\subset B$, we have $A\bigcap B$ is a zero set, therefore *f* is Riemann integrable on the plane. Now, let (ξ,η) be a new set of orthogonal coordinates on the plane.

1°. The ξ-axis is parallel to the *x*-axis. The segment σ_0 is contained in B and parallel to the ξ-axis, but $\int f(x,y_0)\mathrm{d}x$ does not exist (see Figure 34).

2°. The ξ-axis is not parallel to the *x*-axis. The property of the Besicovitch set implies that B contains a segment: $\sigma=\{(\xi,\eta):\eta=\eta_0\ \&\ |\xi|\le 1\}$ that is parallel to the ξ-axis. Notice that $A\bigcap\sigma$ is dense in σ. The discontinuity points of the single variable function $f(\xi,\eta_0)$ would be the whole segment σ, which is not a zero set. Thus, $f(\xi,\eta_0)$ is not Riemann integrable (see Figure 35).

4. Simply Connected

We construct a simple connected Kakeya set with the 4 following steps. It is contained in a circle of radius 1, and its area can be arbitrarily close to 0 ($<\epsilon$).

Step 1. We start with a simple construction. Let Γ be a fixed unit circle and Π be a regular polygon concentric with Γ, having sides of a large **odd number** *Q*. The area of Π can be arbitrarily small as long as the radius of its circumcircle is small enough (see Figure 36).

Lem. Take a vertex of Π, denoted as *C*. Connect the longest diagonals from *C*, denoted as CC_1 and CC_2 (Since *Q* is an odd number, there are two longest diagonals). Extend C_1C and C_2C and they intersect with Γ at *A* and *B* respectively. Up to now, we can slide or rotate a unit segment from C_1A to C_2B continuously. Conduct the same operation to every vertex of Π, we get a star-shape figure, and it is a Kakeya set. We denote triangle CC_1C_2 as Δ, and the triangle outside the polygon as *J*. Denote $K^{(0)}=\Delta\bigcup J$.

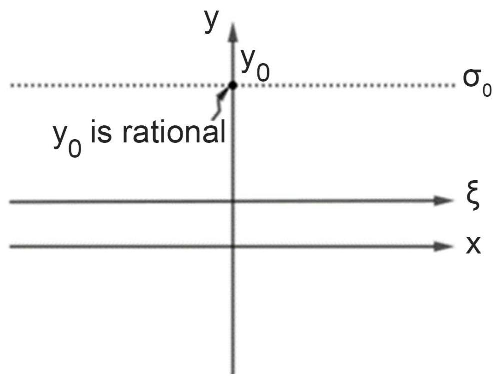

Figure 34.

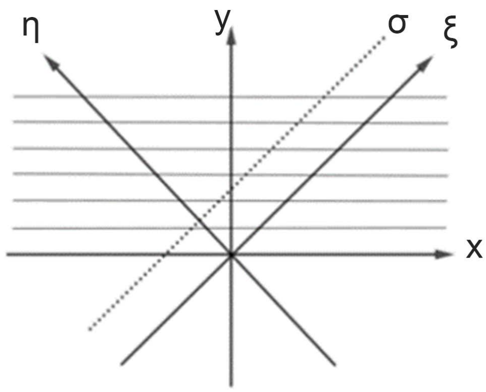

Figure 35.

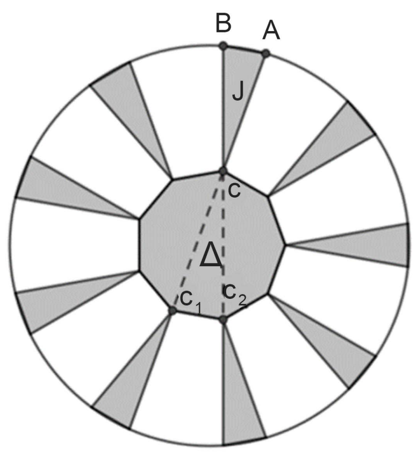

Figure 36.

Now consider the area of Π and those triangles. The area of Π, according to the previous discussion, can be arbitrarily close to 0. The sum of those triangles, however, will be arbitrarily close to $\pi/2$ as long as Q is large enough. (The sum of area of triangles is close to $\frac{1}{2}\cdot\frac{Q}{\pi}\cdot 1\cdot 1\cdot Q=\pi/2$).

Step 2. In this part, we are going to improve $K^{(0)}$ to $K^{(1)}$, which consists of **Tree and Joins** (see Figure 37).

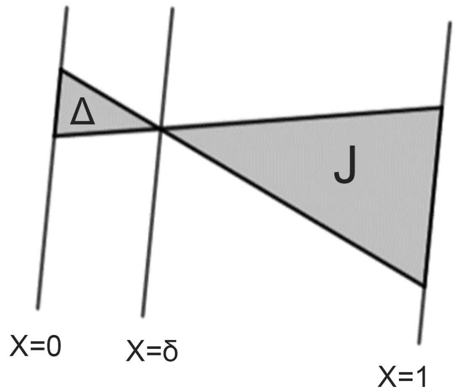

Figure 37.

Tree. Tree is improved from *J*, lies on the right side of $x=0$. We put the $K^{(0)}$ into plane, between two lines $x=0$ and $x=1$. The common vertex of Δ and J is on the line $x=\delta$. We have $\delta+r=1$. Denote Δ as triangle $CA'B'$, where vertex A' and A' are on the line $\{x=0$. Extend $A'C$ and $B'C$, they intersect the line $x=1$ at points A and B respectively.

Choose distinct points on segment $A'B'$ in descending order:

$$C_0=A',C_1,\cdots,C_{m+1}=B'.$$

Choose distinct points on segment AB in ascending order:

$$V_0=A,V_1,V_2,\cdots,V_m=B.$$

The tree is the union of $m+1$ triangles: $\Delta C_iV_iC_{i+1}$, $i=0,1,\cdots,m$.

Joins. Joins lie on the left side of $x=0$. For every C_i, extend $V_{i-1}C_i$ and V_iC_i, intersecting the line $x=-r$ at A_i and B_i respectively. Then we get a triangle $J_i=\Delta C_iA_iB_i$. Joins are the union of m triangles $J_1,\cdots,J_m$.

Denote $K^{(1)}=Tree\bigcup Joins$. Now we consider the area of $K^{(1)}$. Denote $S_J=a$ (S means area), $S_{J_i}=a_i$, then

$$\sum a_i=\frac{1}{2}|A_iB_i|\cdot r=\frac{1}{2}\cdot r\cdot|V_{i-1}V_i|\cdot r=\frac{1}{2}\cdot r^2|AB|=ra.$$

In conclusion, the area of *Joins* is less than the area of *J* after this improvement (see Figure 38).

Step 3. In this part, we make further improvement of the tree, and prove that the area of tree can be arbitrarily close to zero.

First, we put a triangle ΔABC between the line $x=0$ and $x=h'$, one side of which (denoted as AB) is on the line $x=0$. $|AB|=\sigma$. Extend AC and BC, intersecting the line $x=h''$ at V_0 and V_1 respectively ($h''>h'$). The mid-point of AB is M. Connect V_0M and V_1M, intersecting AC and BC at D_1 and D_0 respectively. Shadow the triangle ΔD_1CV_1 and ΔD_0CV_0, and the shadow area is the new additional area. We now calculate the shadow area (see Figure 39).

$$S_{Shadow}=\frac{1}{2}\cdot\frac{(h''-h')^2}{2h''-h'}\cdot\sigma<\frac{(h''-h')^2}{h'}\sigma$$

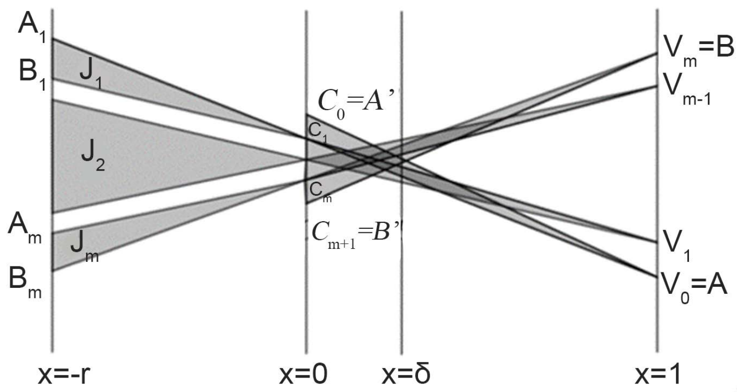

Figure 38.

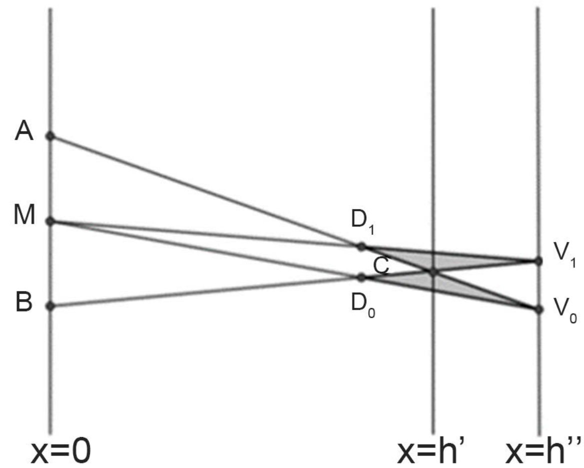

Figure 39.

Now divide the space between $x=\delta$ and $x=1$ into p equal parts. Take $\delta+\frac{r}{p}\cdot i$ and $\delta+\frac{r}{p}\cdot(i+1)$ as h' and h'', employ the improvement above for p times (Let $i=0,1,\cdots,p-1$). The total area of shadow area is no more than $\sum_{i=0}^{p-1}\sigma\cdot\frac{(r/p)^2}{\delta+i\cdot r/p}<\sum_{i=0}^{p-1}\sigma\cdot\frac{(r/p)^2}{\delta}=\frac{\sigma r^2}{p\delta}$. Therefore, $S_{Shadow}\to 0$ as $p\to\infty$ (see Figure 40).

Step 4. Consider every triangle $\Delta A_iB_iC_i$ in the *Joins* of $K^{(1)}$. Denote $S_{\Delta A_iB_iC_i}=a_i$ Extend A_iC_i and B_iC_i, intersecting the line $x=\delta$ at A_i' and B_i' respectively. Let $\Delta C_iA_i'B_i'$ be Δ in $K^{(0)}$, $\Delta A_iB_iC_i$ be J in $K^{(0)}$. Then we make improvement following the Step 2 and Step 3, constructing a small "Tree and Joins". The total area of new joins is no more less than ra_i. Therefore, if we denote the graph after this improvement as $K^{(2)}$, S_{Joins} of $K^{(2)}$ is less than r^2a. Take improvements following the above steps, we can get $K^{(N)}$ with $S_{Joins}<r^Na$ and $S_{Tree}<\epsilon$ (arbitrarily small). In conclusion, the area outside the original triangle Δ can be arbitrarily close to zero. So we construct a Kakeya set with arbitrarily small area.

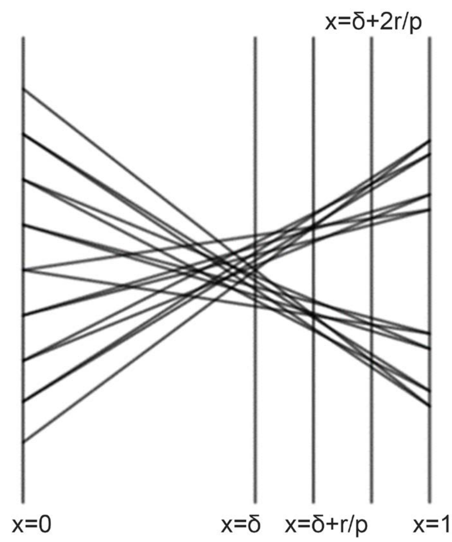

Figure 40.

5. Methodology

The most essential intuition of the Besicovitch set is to cut a basic figure (e.g. a $\frac{1}{6}$ disc) into different pieces and overlap the pieces by conducting parallel technique. In general, we can properly break a Kakeya set into 2 pieces such that a unit line segment can move in each subset. Translating one subset such that it is overlapped on another, generating a new set. After conducting Pal-join trick, it's obvious that a unit line segment can also turn around in the new set, while the overlapping area indicates that the new Kakeya set has a smaller area. Not all Kakeya sets can reduce area by the process. One example is the deltoid curve.

How to define the extent to which a Kakeya set is overlapped? Given a Kakeya set, can we turn it into a set that has no overlapped area? Is there any law that dominates the area of Kakeya set? In order to analyze the question above, probably we need to reconsider the Kakeya set problem in a new way.

Most of our discussions are under the assumption that the slope angle of the segment monotonically increases from 0 to π as the segment turns around in the Kakeya set. The benefit is obvious, since a given slope angle corresponds to a unique position of the segment. The constraint of monotone condition is so strong that it immediately rules out the existence of Pal-joins, which is an essential part of the Besicovitch set shown previously. In an example constructed by Cunningham, a monotone (but not strictly) and simply connected Kakeya set has been proved to exist, since a segment can slide along its direction for any length without costing area. This example will also be ruled out since we require strictly increasing. One natural question is: Is there any possibility that there still exists a Kakeya set with arbitrary small area?

In fact, the following definition and theorem can be extended to the case in which the motion of the segment is not monotone. Due to the limitation of length, they are unable to present.

Intuitively, if the area of a Kakeya set tends to zero during the construction, almost every point in the set would be overlapped infinitely often, which means: given a point in Kakeya set, when the needle turns around, it would sweep through the point infinitely many times. To formalize the above assertion, we need to specify some definitions.

5.1. Definition: Kakeya Dynamic

A Kakeya dynamic is a mapping from the interval to real plane:

$$\phi:[0,\pi]\to\mathbb{R}^2$$

$$t\to\left(x(t),y(t),\theta(t)\right)$$

Subject to the conditions

1) $x(t)$, $x(t)$ and $x(t)$ are C^1.

2) $x(0)=y(0)=0$ and $x(\pi)=1, y(\pi)=0$.

The meaning of the above definition is: the left endpoint of a unit line segment begins at the origin at the beginning. The position of the endpoint is $\left(x(t),y(t)\right)$ given the moving time *t*. The points swept by the segment can be expressed as: $\left(x(t)+u\cos(\theta),y(t)+u\sin(\theta)\right),u\in[0,1]$. Since a Kakeya dynamic is not necessarily monotone, $\theta(t)$ may not be one-one. If it is monotone, it can be simplified to take the following expression.

5.2. Definition: Monotone Kakeya Dynamic

A monotone Kakeya dynamic is a mapping from the interval to the real plane:

$$\phi:[0,\pi]\to\mathbb{R}^2$$

$$\theta\to\left(x(\theta),y(\theta)\right)$$

Subject to the conditions

1) $x(\theta)$ and $y(\theta)$ are C^1.

2) $x(0)=y(0)=0$ and $x(\pi)=1, y(\pi)=0$.

The meaning turns to be: the left endpoint of a unit line segment begins at the origin with zero slope angle. The position of the endpoint is $\left(x(\theta),y(\theta)\right)$ given the slope angle θ. Each point in the Kakeya set takes the form:

$$\left(x(\theta)+t\cos(\theta),y(\theta)+t\sin(\theta)\right).$$

Since the motion is monotone, each slope angle corresponds to unique segment. Thus the above dynamic is well defined.

Since a Kakeya set permits a needle turning around inside it, every Kakeya set corresponds to a dynamic though the dynamic may not be unique.

The above definition enables us study the extent to which a Kakeya set is overlapped. We will soon define the multiplicity to account for it systematically.

5.3. Definition: Dynamic Track

$$\Omega=\left\{\left(x(\theta)+t\cos(\theta),y(\theta)+t\sin(\theta),\theta\right):\theta\in[0,\pi]\right\}\subset\mathbb{R}^3$$

The track of the dynamic lifts the motion process into $\mathbb{R}^3$ space. Through defining a projection mapping, we can deal with the multiplicity of a point in Kakeya set.

5.4. Definition: Projection Mapping

A projection mapping is the inverse of the dynamic which sends each point in dynamic track back to the Kakeya set.

$$\Pi : \Omega \to K$$

$$\left(x(\theta), y(\theta), \theta\right) \to \left(x(\theta), y(\theta)\right)$$

The existence of overlapping guarantees that the projection mapping is an onto but not one-one map. To some extent, it resembles an identification map from Ω to *K*. Now we can precisely measure the extent to which a Kakeya set is overlapped by examining the cardinality of the pre-image of each point in the Kakeya set.

5.5. Definition: Multiplicity

The multiplicity is a mapping from a Kakeya set to the natural numbers:

$$\lambda : K \to \mathbb{N}$$

$$x \to \#\Pi^{-1}(x) = \lambda(x)$$

The multiplicity of a point is defined to be the cardinality of the pre-image of it with respect to the projection mapping. A highly overlapped Kakeya set automatically manifests relatively high multiplicity for each point in it. The area of the set would also become relatively smaller. To precisely express the above thoughts, we define a set that corresponds to a given Kakeya set. The set is named as "unfolded set", the multiplicity of any point in which is one.

5.6. Definition: Unfolded Set

The unfolded set of a Kakeya dynamic is the set expressed in polar coordinates:

$$U = \left\{\left(M(\theta) + u, \theta\right) : u \in [0,1], \theta \in [0,\pi]\right\} \subset \mathbb{R}^2$$

where

$$M(\theta) = \begin{vmatrix} \cos\theta & \dot{x}(\theta) \\ \sin\theta & \dot{y}(\theta) \end{vmatrix}$$

Decompose the motion of the segment into the sliding along direction of the segment and the rotation at the center of a point which lies on the segment or on its extension. Then $M(\theta)$ is exactly the component of the revolution. The unfolded set is aimed to wipe out the sliding motion of the segment and keep the rotation only. We present the example of the deltoid to illustrate the fact precisely (see **Figure 41** & **Figure 42**).

The projection map can induce an onto map from unfolded set to the original Kakeya set.

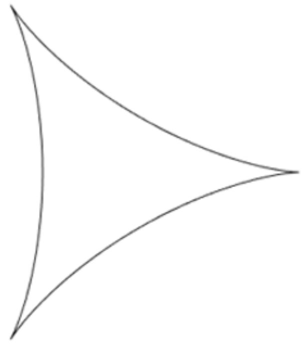

Figure 41.

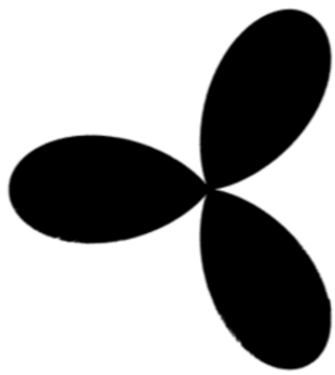

Figure 42.

$$F : U \to K$$
$$\left(M(\theta)+u,\theta\right) \to \left(x(\theta)+u\sin(\theta), y(\theta)+u\cos(\theta)\right)$$

The left side is polar coordinates while the right side takes the form of orthogonal coordinates. F is a C^1 map from U to K.

It would be an immediate result that for any $x \in K$, $\# F^{-1}(x) = \lambda(x)$.

In order to prove our main theorem, we need some lemmas.

1 Lemma. Given a partition of a Kakeya set K:

$$P = \left\{ S_i : S_i \subset K, 1 \le i \le n \right\}$$

Suppose each S_i is simply connected area and the map between K and the unfolded set U induced by projection map is F. Then

$$P' = \left\{ F^{-1}(S_i) : S_i \in P, 1 \le i \le n \right\}$$

is a partition of U.

Proof. To see P' is a partition, we need to show that P' is a cover of U and elements in P' are disjoint, which is equivalent to: for each $p \in U$, there exists unique S_i such that $p \in F^{-1}(S_i)$. For each $q \in U$, since P is a partition of K, there exists $S \in P$ such that $F(q) \in S$, it's obvious that q is contained in $F^{-1}(s_i)$. Thus P' is a covering of U. Now suppose there exist $F^{-1}(S_1)$ and $F^{-1}(S_2)$ that cover a point $p \in U$. We immediately have $F(p) \in S_1 \cap S_2$ which contradicts that $\{S_i\}$ is a partition. □

2 Lemma. There is a closed simply connected set $\{A\}\subset\mathbb{R}^2$. Suppose *f* is a homeomorphism from *A* to *B*, then $B-A\subset\{\langle x,f(x)\rangle : x\in\partial A\}$, where $\langle x,f(x)\rangle$ means the line segment generated by endpoint: *x* and $f(x)$.

Proof. If $B-A$ is empty, there is no point in $B-A$. So we only consider the case where $B-A$ is nonempty.

If $p\in\partial B$. Since the homeomorphism from *A* to *B* induces a homeomorphism from ∂A to ∂B, there exists unique preimage of *p*, denoted by p', and $p'\in\partial A$. It's obvious that $p\in\langle p',f(p')\rangle$.

Now we suppose that $p\in int(B)$. *A* is simply connected, f is homeomorphism, so *B* is simply connected and ∂B is a Jordan curve. Moreover, notice that $p\notin A$, so we have: *p* is encompassed by ∂B while it's not done by ∂A (see Figure 43). This drastic difference is the point of the proof.

Suppose the perimeter of ∂A is *l*. Fix a point $O\in\partial A$. A point *Q* is moving along ∂A, which is a closed curve, from the beginning position *O* until it back to that original point. Denote the travelled distance of *Q* to be *x*, then the slope angle of line *PQ* would be a continuous function of *x*, denoting $\gamma(x)$.

The image $f(Q)=:Q'$ is also going a full round along ∂B as *Q* travels a round along ∂A since f is a homeomorphism. The slope angle of PQ' is also a continuous function of *x*, denoting $\theta(x)$. Without loss of generality, we define that $\gamma(0)=0$, which can be done by rotating *x* axis. Then, according to the above fact, we have: $\gamma(l)=2\pi$, and $\theta(0)=\theta(l)$.

To see how the above definition contributes to the proof, notice that there eixsts x_0 such that $p\in\langle x_0,f(x_0)\rangle$ if and only if there exists x_0 such that $\gamma(x_0)+\theta(x_0)=(2k+1)\pi, k\in\mathbb{Z}$ (see Figure 43).

Since there exists $k_0\in\mathbb{Z}$ such that $(2k_0+1)\pi-\theta(0)\in[0,2\pi]$, we immediately have: $\gamma(0)+\theta(0)-(2k_0+1)\pi\le 0$ and $\gamma(l)+\theta(l)-(2k_0+1)\pi\le 0$. The existence theorem of zero point of continuous function claims that there exists $x_0\in[0,l]$ such that $\gamma(x_0)+\theta(x_0)=(2k_0+1)\pi$ (see Figure 44). □

3 Lemma. There is a closed simply connected set $\{A\}\subset\mathbb{R}^2$. Suppose *f* is a homeomorphism from *A* to *B* ($B\subset\mathbb{R}^2$). Then $B\subset A\cup D$, where $D:=\cup M_{\|f(x)-x\|}(x)$, *x* is boundary point of *A*.

Proof. Suppose not, then there exist a point $p\in B$ such that $p\notin A\cup D$, which also means that $p\in B-A$. According to lemma2, there exists $x_0\in\partial A$ such that $p\in\langle x_0,f(x_0)\rangle$. Hence it's obvious that $p\in M_{\|f(x_0)-x_0\|}(x_0)$. □

4 Lemma. There is a family of closed simply connected measurable set $\{A_n\}\subset\mathbb{R}^2$ with Hausdorff dimension 2, the boundary of A_n has Hausdorff dimension 1. The diameter d_n of A_n tends to 0 as $n\to\infty$. Suppose f_n is a sequence of homeomorphisms from A_n to B_n ($B_n\subset\mathbb{R}^2$) such that $\forall x_n\in A_n$, $\frac{\|f(x_n)-x_n\|}{d_n}\to 0$ as $n\to\infty$. Then $\frac{\||A_n|-|B_n|\|}{|A_n|}\to 0$ as $n\to\infty$.

Proof. For each *n*, Since f_n is onto map, $H_n:=\{O_{x_n}:O_{x_n}$ is $\|f(x_n)-x_n\|$-neighborhood of x_n, $\forall x_n\in A_n\}$ covers B_n. Denote:

$$D_n:=\cup O_{x_n}\ \text{ if } x_n \text{ is boundary point of } A_n.$$

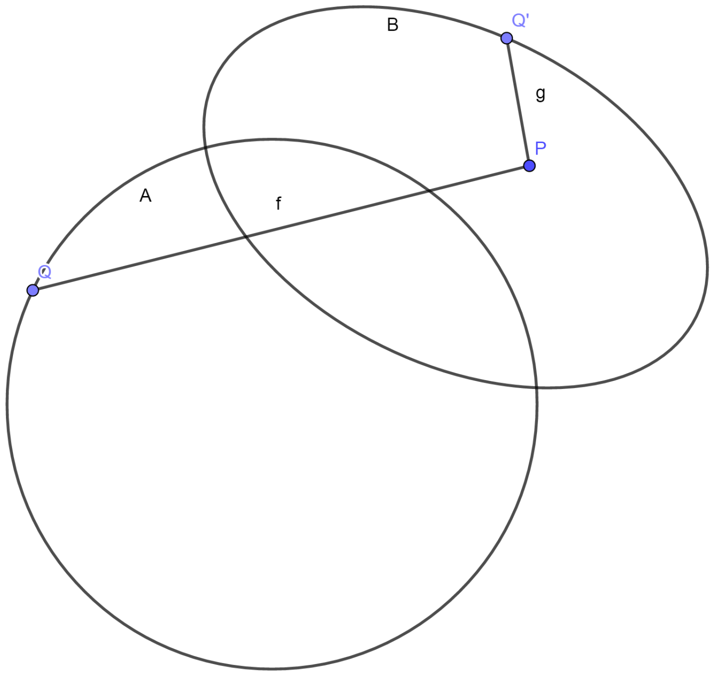

Figure 43.

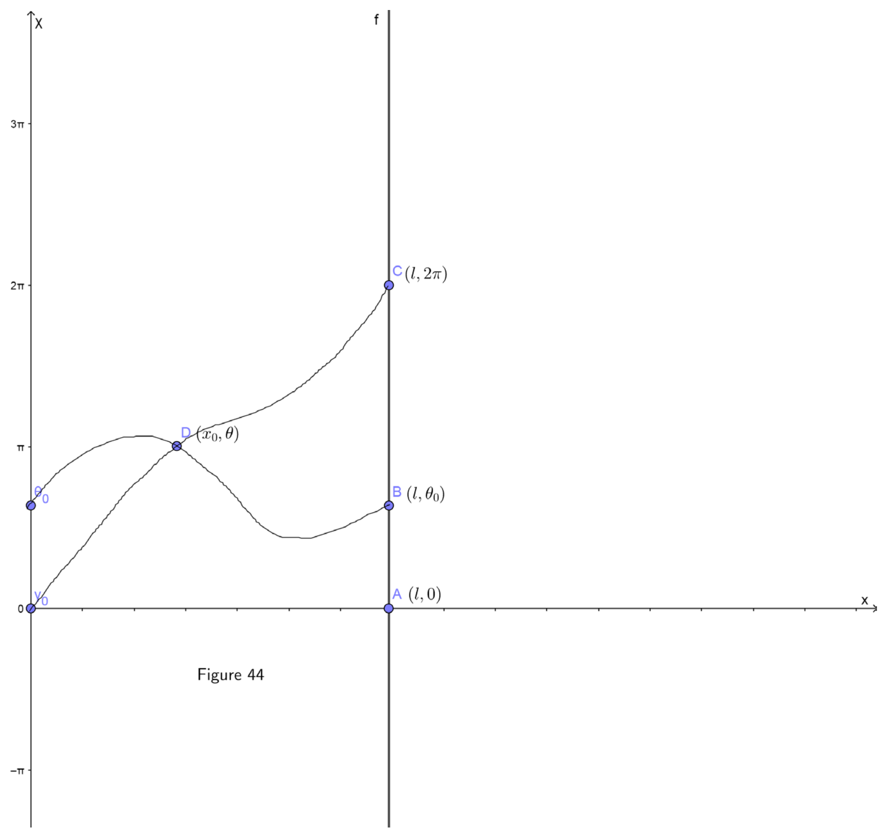

Figure 44.

According to Lemma 3, $B_n \subset A_n \cup D_n$, we have $|B_n| \le |A_n| + |D_n|$, *i.e.* $|A_n| - |B_n| \ge -|D_n|$.

On the other hand, f_n being a homeomorphism implies that f_n is invertible, hence $G_n := \{ O_{x_n} : O_{x_n}$ is $\left\| x_n - f^{-1}(x_n) \right\|$-neighborhood of x_n, $\forall x_n \in B_n \}$ covers A_n.

Similarly, we can denote:

$$D_n' := \bigcup O_{x_n} \text{ if } x_n \text{ is boundary point of } B_n.$$

And we have $|A_n| - |B_n| \le |D_n'|$. In one word, $-|D_n| \le |A_n| - |B_n| \le |D_n'|$.

In order to show $\frac{\left\| A_n \right| - \left| B_n \right\|}{|A_n|} \to 0$ as $n \to \infty$, it suffices to show that $\frac{|D_n|}{|A_n|} \to 0$ as $n \to \infty$ and $\frac{|D_n'|}{|A_n|} \to 0$ as $n \to \infty$.

Notice that $|D_n|$ is a cover of the boundary of A_n with open discs. The Hausdorff dimension of boundary being 1 implies that $\frac{|D_n|}{d_n}$ tends to a finite number as $n \to \infty$. Similarly, we have $\frac{|D_n'|}{d_n}$ tends to a finite number as $n \to \infty$.

Also, Notice that the Hausdorff dimension of A_n being 2 implies that $\frac{|A_n|}{d_n^2}$ tends to a finite number as $n \to \infty$.

In conclusion, $\frac{|D_n|}{|A_n||d_n|}$ tends to a finite number as $n \to \infty$ and $\frac{|D_n'|}{|A_n||d_n|}$ tends to a finite number as $n \to \infty$, so we have $\frac{\left\| A_n \right| - \left| B_n \right\|}{|A_n|} \to 0$ as $n \to \infty$. □

Under the C^1 setting of the Kakeya dynamic, the following lemma shows that the multiplicity of a Kakeya dynamic is a.e. bounded. This finite condition is useful.

5 Lemma. The multiplicity $\lambda(p)$ is almost everywhere bounded in the Kakeya set *K*.

Proof. Suppose not, then there exists an open neighborhood $O \subset K$ such that $\forall x \in O$, the multiplicity of *x*, denoting $\lambda(x)$, is infinity. *O* is open implies that $\exists$ a closed segment $\langle x, y \rangle \subset O$. For each point *Z* on $\langle x, y \rangle$, there are infinite many $\theta_i \in [0, \pi]$ such that their corresponding unit segments go through *Z*. By extending the unit segment and $\langle x, y \rangle$ to be real line and taking the intersection, we naturally induce a mapping *h* from $[0, \pi]$ to $\mathbb{R}$ if we give a parameterization to the line determined by $\langle x, y \rangle$. Since the intersection of 2 lines in plane exists unless they are parallel, the monotone condition implies that the mapping *h* is C^1 continuous except for a point. Denote by $[a, b]$ the interval corresponding to $\langle x, y \rangle$ under the parameterization. Then *h* should satisfy: for each $s \in [a, b]$, $f^{-1}(s)$ is a infinite set in the interval $[0, \pi]$, according to Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{u_i(s)\} \to u_0(s)$ as a subset of $f^{-1}(s)$. The continuity of *f* implies that $f(u_i(s)) \to f(u_0(s))$. Since $f(u_i(s)) = s$, we have $f(u_0(s)) = s$ and $f'(u_0(s)) = 0$. We can simply denote by $U_0(s)$ the collection of all such cluster point like $u_0(s)$.

Next, we claim that for each $s \in [a,b]$ there exist a $u_0(s) \in U_0(s)$ and an open neighborhood O_s of $u_0(s)$ such that $\forall s \neq v, U(v) \cap O_s$ is empty, if this property holds, the closed interval $[0,\pi]$ is covered by a family of uncountable disjoint open sets $\{O_s\}, s \in [a,b]$, a contradiction.

Suppose there exist a $u_0(s_0)$ such that any open neighborhood of $u_0(s_0)$ intersects with some $U(v)$, then we can construct a sequence $\{s_i\} \to s_0 \quad (s_i \neq s_0)$ with $u_0(s_i) \to u_0(s_0)$ by compressing that open neighborhood. Since for each s_i, $f'(s_i) = 0$, f is constant and $s_i = s_0$, a contradiction. □

Lemma 5 claims that for almost every point $p \in K$ with multiplicity $\lambda(p)$, there exist finite segments $l_1, l_2, \cdots, l_{\lambda(p)}$ containing p. Without loss of generality, we assume the slope angle of l_i is θ_i and $\theta_i > \theta_j$ if $i > j$. We can express coordinate of the point p in $\lambda(p)$ ways, *i.e.*

$$p = \left(x(\theta_i) + u_i \cos(\theta_i), y(\theta_i) + u_i \sin(\theta_i)\right).$$

Suppose F is the mapping from the unfolded set to the Kakeya set constructed previously. $F^{-1}(p)$ takes the form of $\left\{\left(M(\theta_i) + u_i, \theta_i\right) : 1 \leq i \leq \lambda(p)\right\}$, which is a collection of $\lambda(p)$ distinct points in U.

Take a partition of a Kakeya set K:

$$P = \{S_i : S_i \subset K, S_i \text{ is simply connected}, 1 \leq i \leq n\}$$

Given that the diameter of S_k is small enough, $F^{-1}(p)$ induce $\lambda(p)$ number of homeomorphism f_k between s_k and a neighborhood of $\left(M(\theta_i) + u_i, \theta_i\right)$: V_i, with $F(V_i) = S_i$.

Now we consider a particular homeomorphism, say f_k:

$$f_k : S_k \to V_i$$

$$\left(x(\theta_i) + u\cos(\theta_i), y(\theta_i) + u\sin(\theta_i)\right) \to \left(M(\theta_i) + u, \theta_i\right)$$

The below figure (**Figure 45**) shows the pattern of the motion in $\Delta\theta$ neighborhood, where the velocity vector $(\dot{x}(\theta), \dot{y}(\theta))$ is decomposed into the normal component and radial component.

Computation shows that $M(\theta)$ is the distance between the rotation center and the endpoint. Take a point q in S_k. Triangle inequality implies that $\|f_k(q) - q\| \leq \cot\alpha(\Delta\theta)^2$, while the diameter of $S_k \geq u(\Delta\theta)$. The mesh of S_k is at least: $L\Delta\theta$, where L is the distance between q and the rotation center. Thus

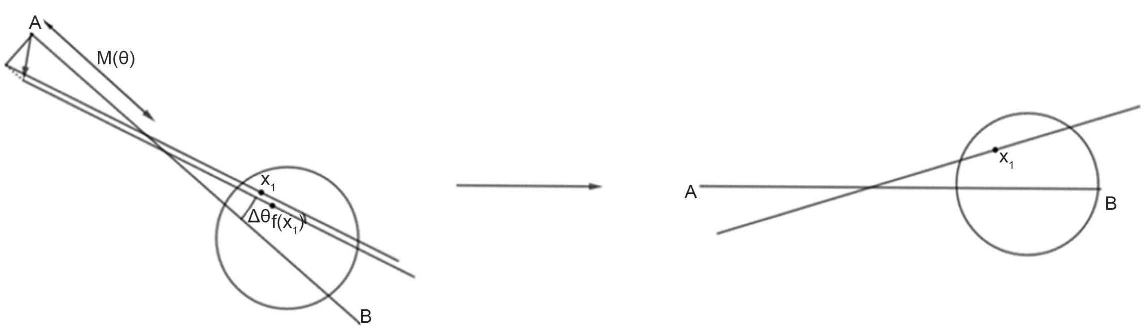

Figure 45.

$\frac{\|f_k(q)-q\|}{d_n} \to 0$ as $n \to 0$, so we can apply the lemma1, lemma2, lemma3 and lemma4 to get theorem 6.

6 Theorem. The integral of multiplicity on Kakeya set K is equal to the area of unfolded set U:

$$\iint_K \lambda(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_U 1\mathrm{d}x\mathrm{d}y$$

Proof. Given a series of partition of Kakeya set K: $P_n = \{S_i : S_i \subset K, 1 \le i \le n\}$ with mesh tends to 0 as $n \to \infty$.

The Riemann sum of $\iint_K \lambda(x,y)\,\mathrm{d}x\mathrm{d}y$ is $\sum_i \lambda(x_i,y_i)|s_i|$. The Riemann sum of $\iint_U 1\mathrm{d}x\mathrm{d}y$ is $\sum_i \left|F^{-1}(S_i)\right|$. The difference between them is:

$$\sum_i \sum_{k=1}^{\lambda(x_i,y_i)} |S_i| - |f_k(S_i)|.$$

From the lemma above, $\frac{\||S_i| - |f_k(S_i)|\|}{S_i} \to 0$ as $n \to 0$. Moreover, $\lambda(x_i,y_i)$ is almost everywhere bounded on K, we have $\frac{\lambda(x_i,y_i)\left(\||S_i|-|f_k(S_i)|\|\right)}{|S_i|} \to 0$ as $n \to \infty$. So we have $\frac{\sum_i \sum_{k=1}^{\lambda(x_i,y_i)} |S_i| - |f_k(S_i)|}{\sum_i |S_i|} \to 0$ as $n \to \infty$.

Notice that K is Riemann integrable since $x(\theta), y(\theta)$ are continuously differentiable functions. The sum $\sum_i |S_i|$ is finite. So

$$\frac{\sum_i \sum_{k=1}^{\lambda(x_i,y_i)} |S_i| - |f_k(S_i)|}{\sum_i |S_i|} \to 0$$

as $n \to \infty$ immediately implies: $\sum_i \sum_{k=1}^{\lambda(x_i,y_i)} |S_i| - |f_k(S_i)| \to 0$ as $n \to \infty$. Thus we identify the Riemann sum above. □

The unfolded set can be regarded as the area swept by a moving unit segment, while the segment must intersect with a fixed point during the whole motion process. It's obvious that under such constraint, the minimum area of the unfolded set is $\frac{1}{2}\pi$, obtained by the disc with diameter of 1. We have the following theorem.

7 Theorem. Given a monotone Kakeya dynamic, the integral of multiplicity on Kakeya set is greater than $\frac{1}{2}\pi$.

$$\iint_K \lambda(x,y)\,\mathrm{d}x\mathrm{d}y \ge \frac{1}{2}\pi$$

It's a direct deduction that if the area of a sequence of Kakeya sets tends to zero, the multiplicity of points in Kakeya sets must tend to infinity.

The estimation of multiplicity can fully explain how the Besicovitch set and the Cunningham simply connected set achieve an arbitrary small area, they are highly overlapped while the trick of the overlapping is not the same. The Besicovitch one used Pal-join which can translate the segment at any direction. Actually, the motion of the segment also implies that any Pal-join will spoil the monotone assumption. In the case of Cunningham, the trick becomes: a line segment can slide along its direction which will hold the property of simply connected and cost no area at the same time. Again, the motion is not strictly monotone if any slide takes place. What would be the solution if we require that the dynamic must be monotone? We have the following conjecture.

Conjecture. The optimal solution is taken by deltoid when we minimize the area of strictly monotone Kakeya dynamic.

6. Derivation of Kakeya Problem

Similarly, we can ask the minimum measure that allows a disc or something more general to turn around in some more general spaces (e.g. *n*-sphere, real projective plane or so). Currently, it has been proved that there exists Kakeya set of arbitrary small measure in sphere. The spherical Kakeya problem reads: Instead of a plane, the rotation takes place on the surface of a unit sphere and an arc of great circle substitutes the needle. Cunningham showed that for the arc length *a*: $0 \le a \le \pi$, the lower bound of Kakeya set area is still 0.

In the range of dimension theory, it was shown by Davies that, in Euclidean spaces, even though Kakeya sets had zero area, they were still necessarily two-dimensional, which led to an analogous conjecture in higher dimensions: the Hausdorff dimension of a Kakeya set is *n* dimensional Euclidean space. That is the Kakeya conjecture.

1 Conjecture. A Besicovitch set in $\mathbb{R}^n$ has Minkowski and Hausdorff dimension n.

In an algebraic field, there is an analogous conjecture. The Kakeya set in a finite field F^n is a finite point set *K* such that for any $v \in F^n$, there exists a point $x \in F^n$ such that the whole line: $\{x + tv : t \in F\}$ is contained in *K*. the Kakeya conjecture in Euclidean space takes the form of the finite field conjecture.

2 Conjecture. Suppose *F* is finite field and $E \subset F^n$ is a Kakeya set. Then *E* has cardinality at least $c_n |F|^n$, where $c_n > 0$ depends only on *n*.

The finite field Kakeya conjecture was proved by Zeev Dvir in 2008. The method is to combine the Kakeya set with a polynomial which vanishes on that Kakeya set. The feature of Kakeya set implies that any polynomial of degree at most $|F|-1$ which vanishes on a Kakeya set E must be the null polynomial. Then, the cardinality of a Kakeya set must exceed the dimension of the vector space: $\{ P \in F[x_1, \cdots, x_n]$: the degree of *P* is less than $|F| \}$, which is $C_n^{|F|+n-1}$.

7. Summary

By and large, this thesis summarizes the classical results of the Kakeya needle problem. The Kakeya needle problem asks for the minimum area in which a line segment can turn around. The first attempt is to search for a solution in a convex set. The minimum convex set turns out to be triangle with height of 1. As for non-convex sets, mathematicians believed that deltoid was the optimal solution but it was left unproved until Besicovitch found a Kakeya set of arbitrary small area.

We presented 3 methods in constructing Besicovitch sets. The second construction is simpler in the sense of estimating the area of each "horn". The last one makes some minor changes: the unfolded set changes from a triangle to a sector. After the construction, we presented the original intuition of Besicovitch sets, which is connected to the existence of iterated integrals in the real plane.

In 1971, Cunningham provided a simply connected Kakeya set of area less than any real number, which broke through the belief that every simply connected Kakeya set had area greater than the Bloom-Schoenberg number. We presented the process of constructing a Cunningham Kakeya set but omitted the proof of simply connectedness.

The Methodology part generalized the trick took place in the Besicovitch set. First, through paraphrasing the Kakeya problem from a dynamic point of view, we defined the multiplicity of a Kakeya set. Then we provided a theorem to explain the fact that if a Kakeya set can achieve arbitrary small area, the set must be highly overlapped. In modern analysis, the concept of multiplicity (not exactly the same though) is also used in estimating the bound of Hausdorff dimension when mathematicians discuss Kakeya maximal functions. The end of this section put forward a conjecture of the constraint under which a deltoid becomes the optimal solution of the Kakeya problem.

There are many forms of the Kakeya needle problem. Many of them were put forward in a relatively modern way. The most important unsolved problem may be the Kakeya conjecture. The algebraic analogue is known as finite-field Kakeya conjecture. The Kakeya conjecture is connected with Fourier analysis, additive combinatorics and partial differential equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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