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Diagrammatic Approach for Investigating Two Dimensional Elastic Collisions in Momentum Space I: Newtonian Mechanics

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Abstract

We present the usefulness of the diagrammatic approach for analyzing two dimensional elastic collision in momentum space. In the mechanics course, we have two major purposes of studying the collision problems. One is that we have to obtain velocities of the two particles after the collision from initial velocities by using conservation laws of momentum and energy. The other is that we have to study two ways of looking collisions, *i.e.* laboratory system and center-of-mass system. For those two major purposes, we propose the diagrammatic technique. We draw two circles. One is for the center-of-mass system and the other is for the laboratory system. Drawing these two circles accomplish two major purposes. This diagrammatic technique can help us understand the collision problems quantitatively and qualitatively.

Keywords

Two Dimensional Elastic Collision, Momentum Space, Laboratory System, Center-of-Mass System, Newtonian Mechanics

1. Introduction

Collisions are of importance in physics. Especially for small world such as atoms or nuclei, scattering is the crucial technique to investigate their nature. Before studying the scattering theory of quantum mechanics, we had better to get familiar with collisions in classical mechanics.

We have two main themes for studying the collision problems [1] [2]. One is that we have to obtain velocities of the two particles after the collision from initial velocities by using conservation laws of momentum and energy. Since we have three equations from conservation law and four unknowns, one parameter out of four should be fixed according to the given collision problems. The other is that we have to study two ways of looking collisions, *i.e.* laboratory system and center-of-mass system. We need to convert between the two systems.

In order to achieve those two themes, the diagrammatic technique gives the powerful tool. For the collisions in one dimension, the mass-momentum diagram with mass m along the vertical axis and momentum p represented on the horizontal axis is the useful technique [3]. We obtain the whole story of the collision in one dimension from the single (m, p)-diagram.

When we apply the mass-momentum diagram to the elastic collisions in two dimensions, three dimensional space (m, p_x, p_y) is needed. However, in this case, the projection onto the (p_x, p_y)-plane is sufficient [4]. For Newtonian mechanics [5], we draw a circle in the two dimensional momentum space and also draw arrows of the momentum of the colliding particles into the circle. This is a way of looking collisions from laboratory system. In this article, we add one more circle. This is from the point of view of the center-of-mass system. These two circles give all information for two dimensional elastic collision problems.

This paper is organized in the following way. In Section 2, we recall two dimensional elastic collisions with equations. In Section 3, we show the diagrammatic approach for two dimensional elastic collision in order. First, we draw a circle for the center-of-mass system. Then we add to draw one more circle to obtain the momentum after the collision in the laboratory system. In Section 4, we investigate the special case where the target particle is at rest before the collision. Section 5 is devoted to a conclusion.

2. Elastic Collision between Two Particles in Two Dimensions

Let us recall the treatise of the two dimensional elastic collision with equations for later use. **Figure 1** shows the collisions from the point of view in laboratory and center-of-mass systems and also show the notation which we use in this article. The projectile A has mass m_A and velocity $\boldsymbol{v}_A$ and the target B has mass m_B and velocity $\boldsymbol{v}_B$ before the collision. The momenta are given by $\boldsymbol{p}_A = m_A\boldsymbol{v}_A$ and $\boldsymbol{p}_B = m_B\boldsymbol{v}_B$. They are known parameters or initial conditions of the collision. Usually, the velocities $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$ are supposed to be in the

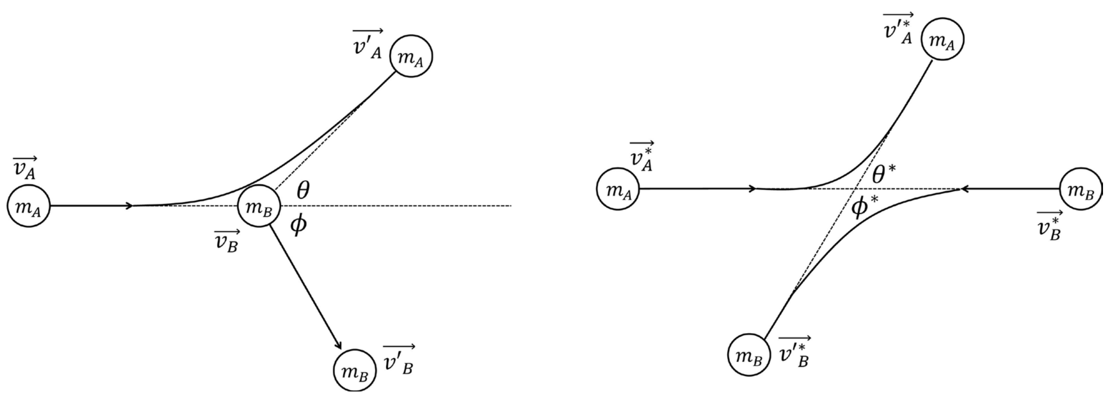

Figure 1. Left: Collisions in the laboratory system. Right:Collisions in the center-of-mass system.

same direction and are set along the x-axis in this article. The velocities after the collision are distinguished by the primes. And the asterisk is attached to the parameters in the center-of-mass system.

We have to obtain the four parameters $\left(v'_A, v'_B, \theta, \phi\right)$ after the collision in the laboratory system. However, we have only three equations, *i.e.* energy conservation and two components of momentum conservations. So, we have to fix one parameter out of four. We investigate the collision in the following way.

2.1. Velocities of the Center-of-Mass

We write down the conservation of momentum for two systems:

$$m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B = m_A \boldsymbol{v}'_A + m_B \boldsymbol{v}'_B \quad \text{for laboratory system,} \tag{1}$$

$$m_A \boldsymbol{v}^*_A + m_B \boldsymbol{v}^*_B = m_A \boldsymbol{v}'^*_A + m_B \boldsymbol{v}'^*_B = \boldsymbol{0} \quad \text{for center-of-mass system.} \tag{2}$$

Let $\boldsymbol{V}$ be the velocity of the center-of-mass. The relations between $\boldsymbol{V}$ and the velocities of the particles in two systems are as follows:

$$\boldsymbol{v}^*_A = \boldsymbol{v}_A - \boldsymbol{V}, \quad \boldsymbol{v}^*_B = \boldsymbol{v}_B - \boldsymbol{V}. \tag{3}$$

Substituting Equation (3) into Equation (2), we obtain

$$\boldsymbol{V} = \frac{m_A \boldsymbol{v}_A + m_B \boldsymbol{v}_B}{m_A + m_B} = \frac{\boldsymbol{p}_A + \boldsymbol{p}_B}{m_A + m_B}, \tag{4}$$

which remains the same before and after collision.

2.2. Velocities and Momenta before Collision in Center-of-Mass System

Substituting Equation (4) into Equation (3), we obtain the velocities in the center-of-mass system

$$\boldsymbol{v}^*_A = +\frac{m_B}{m_A + m_B}\left(\boldsymbol{v}_A - \boldsymbol{v}_B\right), \quad \boldsymbol{v}^*_B = -\frac{m_A}{m_A + m_B}\left(\boldsymbol{v}_A - \boldsymbol{v}_B\right), \tag{5}$$

and the momenta

$$\boldsymbol{p}^*_A = +\frac{m_A m_B}{m_A + m_B}\left(\boldsymbol{v}_A - \boldsymbol{v}_B\right), \quad \boldsymbol{p}^*_B = -\frac{m_A m_B}{m_A + m_B}\left(\boldsymbol{v}_A - \boldsymbol{v}_B\right). \tag{6}$$

We clearly see that these expressions of the momentum satisfy the relation in Equation (2). Note that $\frac{m_A m_B}{m_A + m_B}$ is a reduced mass and $\boldsymbol{v}_A - \boldsymbol{v}_B$ $\left(v_A > v_B\right)$ is a relative velocity before the collision.

2.3. Velocities and Momenta after Collision in Center-of-Mass System

We write energy conservations for two systems:

$$\frac{m_A}{2}\left(\boldsymbol{v}_A\right)^2 + \frac{m_B}{2}\left(\boldsymbol{v}_B\right)^2 = \frac{m_A}{2}\left(\boldsymbol{v}'_A\right)^2 + \frac{m_B}{2}\left(\boldsymbol{v}'_B\right)^2 \quad \text{for laboratory system,} \tag{7}$$

$$\frac{m_A}{2}\left(\boldsymbol{v}^*_A\right)^2 + \frac{m_B}{2}\left(\boldsymbol{v}^*_B\right)^2 = \frac{m_A}{2}\left(\boldsymbol{v}'^*_A\right)^2 + \frac{m_B}{2}\left(\boldsymbol{v}'^*_B\right)^2 \quad \text{for center-of-mass system.} \tag{8}$$

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From the conservations of momentum in Equation (2), we obtain the following relations:

$$\boldsymbol{v}_B^* = -\frac{m_A}{m_B}\boldsymbol{v}_A^*, \quad \boldsymbol{v}_B'^* = -\frac{m_A}{m_B}\boldsymbol{v}_A'^*. \tag{9}$$

Substituting these relations into Equation (8), we obtain

$$v_A'^* = v_A^*, \quad v_B'^* = v_B^*. \tag{10}$$

This means that the velocities (and also momenta) of the two particles stay in magnitude before and after the collision in the center-of-mass system. Thus the collision simply rotates the velocities. However, the angle of the rotation cannot be determined from the conservations of momentum and energy because we have four unknowns and only three equations: the energy and the two components of momentum conservations. Namely, there is an infinite number of possible final states of outgoing particles in an elastic collision in two dimensions. Let $\boldsymbol{n}^*$ be a unit vector in the direction of the velocity $\boldsymbol{v}_A'^*$ of the projectile A after the collision in the center-of-mass system. The scattering angle θ^* of the right figure in **Figure 1** is related by $\boldsymbol{n}^* = \left(\cos\theta^*, \sin\theta^*\right)$. Accordingly, the velocities after the collision in the center-of-mass system are written by

$$\boldsymbol{v}_A'^* = +\frac{m_B}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^*, \quad \boldsymbol{v}_B'^* = -\frac{m_A}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^*, \tag{11}$$

and the momenta are given by

$$\boldsymbol{p}_A'^* = +\frac{m_A m_B}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^*, \quad \boldsymbol{p}_B'^* = -\frac{m_A m_B}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^*. \tag{12}$$

Again, these expressions of the momentum satisfy the relation in Equation (2).

2.4. Velocities and Momenta after Collision in Laboratory System

In order to return to the laboratory system, we must add Equation (11) to the velocity of the center-of-mass $\boldsymbol{V}$ in Equation (4). We obtain velocities after the collision in the laboratory system,

$$\boldsymbol{v}_A' = \boldsymbol{v}_A'^* + \boldsymbol{V}, \quad \boldsymbol{v}_B' = \boldsymbol{v}_B'^* + \boldsymbol{V}, \tag{13}$$

and momenta

$$\boldsymbol{p}_A' = \boldsymbol{p}_A'^* + m_A\boldsymbol{V} = +\frac{m_A m_B}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^* + \frac{m_A}{m_A + m_B}\left(\boldsymbol{p}_A + \boldsymbol{p}_B\right), \tag{14}$$

$$\boldsymbol{p}_B' = \boldsymbol{p}_B'^* + m_B\boldsymbol{V} = -\frac{m_A m_B}{m_A + m_B}\left|\boldsymbol{v}_A - \boldsymbol{v}_B\right|\boldsymbol{n}^* + \frac{m_B}{m_A + m_B}\left(\boldsymbol{p}_A + \boldsymbol{p}_B\right). \tag{15}$$

Note that the sum of both sides clearly shows the momentum conservation $\boldsymbol{p}_A' + \boldsymbol{p}_B' = \boldsymbol{p}_A + \boldsymbol{p}_B$ of Equation (1) in the laboratory system.

Let $\boldsymbol{p}_A' = \left(p_{Ax}', p_{Ay}'\right)$ be the x, y-components of the momentum of the projectile A in Equation (14). We write down explicitly as follows:

$$p'_{Ax} = \frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B| \cos\theta^* + m_A V, \tag{16}$$

$$p'_{Ay} = \frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B| \sin\theta^*, \tag{17}$$

where we note that the velocity $\boldsymbol{V}$ has the x-component only due to Equation (4). From these equations and the relation $\cos^2\theta^* + \sin^2\theta^* = 1$, we obtain

$$\left(\frac{p'_{Ax} - m_A V}{\frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B|} \right)^2 + \left(\frac{p'_{Ay}}{\frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B|} \right)^2 = 1. \tag{18}$$

This indicates the circle in momentum space, centered at $(m_A V, 0)$ with its radius $p_A^* = p_B^* = \frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B|$. These quantities are uniquely determined by the initial condition of the collision.

Let us consider the case where the target particle is at rest before the collision. Setting $\boldsymbol{v}_B = \boldsymbol{0}$ in Equation (18), we obtain

$$\left(\frac{p'_{Ax} - \frac{m_A}{m_A + m_B} p_A}{\frac{m_B}{m_A + m_B} p_A} \right)^2 + \left(\frac{p'_{Ay}}{\frac{m_B}{m_A + m_B} p_A} \right)^2 = 1. \tag{19}$$

Moreover, the case of which two particles have equal mass $m_A = m_B$ becomes more simple:

$$\left(\frac{p'_{Ax} - \frac{p_A}{2}}{\frac{p_A}{2}} \right)^2 + \left(\frac{p'_{Ay}}{\frac{p_A}{2}} \right)^2 = 1. \tag{20}$$

These equations of a circle will appear in the next sections.

3. Diagrammatic Technique

In this section, we deduce all relations, which we recalled in the former section, from the diagram in two dimensional momentum space.

3.1. Center-of-Mass System

Firstly, we draw a circle whose radius is $p_A^* = p_B^*$ in Equation (6), as depicted in **Figure 2**. This dashed circle centered at O describes the collision in the center-of-mass system. We draw dashed arrows of the momenta before the collision from Equation (6)

$$\boldsymbol{OA} = \boldsymbol{p}_A^* = +\frac{m_A m_B}{m_A + m_B} (\boldsymbol{v}_A - \boldsymbol{v}_B), \quad \boldsymbol{OB} = \boldsymbol{p}_B^* = -\boldsymbol{p}_A^*, \tag{21}$$

and after the collision from Equation (12)

$$\boldsymbol{OC} = \boldsymbol{p}_A'^* = +\frac{m_A m_B}{m_A + m_B} |\boldsymbol{v}_A - \boldsymbol{v}_B| \boldsymbol{n}^*, \quad \boldsymbol{OD} = \boldsymbol{p}_B'^* = -\boldsymbol{p}_A'^*, \tag{22}$$

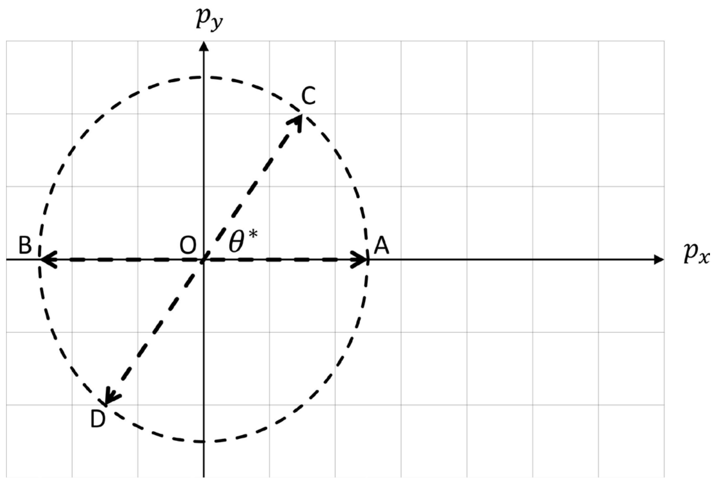

Figure 2. Collision in the center-of-mass system. The figure shows the case that the projectile A has mass $m_A = 3$ and velocity $v_A = \frac{8}{3}$ and the target B has mass $m_A = 4$ and velocity $v_B = -\frac{1}{4}$. The vectors $\boldsymbol{OA} = \boldsymbol{p}_A^* = (5,0) = -\boldsymbol{OB} = -\boldsymbol{p}_B^*$ are the momenta of incident particles before the collision. The vector $\boldsymbol{OC} = \boldsymbol{p}_A'^*$ and $\boldsymbol{OD} = \boldsymbol{p}_B'^*$ are the momentum of outgoing particles after collision. The angle $\angle COA = \theta^*$ cannot be determined by the conservation laws only. We fix it according to the given collision problem.

where $\boldsymbol{n}^* = (\cos\theta^*, \sin\theta^*)$ or $\theta^* = \angle COA$ is determined according to what we are asked in the collision problems. Since the scattering angle θ^* cannot be determined by the conservations of momentum and energy, the point C lies anywhere on this circle and the point D is opposite side against the point C on the circle.

3.2. Laboratory System

Next, as shown in **Figure 3**, we determine the point E on the p_x-axis so that $\boldsymbol{OE} = m_A \boldsymbol{V}$. We draw another circle centered at E whose radius is the same as the dashed circle centered at O. This circle centered at E is appeared in the book of Landau and Lifshitz [5], and its equation is written by Equation (18). This circle describes the laboratory system as explained below.

As shown in **Figure 3**, we draw a broken line from the point C in parallel to the p_x-axis until the broken line intersects with the circle centered at E. We call this point of intersection as F. Note that $\boldsymbol{OC} = \boldsymbol{EF}$ is always satisfied and it means $\angle COA = \angle FEA = \theta^*$. Then, the vector $\boldsymbol{OF} = \boldsymbol{p}_A'$ shows the momentum of the projectile A after the collision in the laboratory system. The tips of $\boldsymbol{OF} = \boldsymbol{p}_A'$ is always on the circle centered at E. The angle $\angle FOE = \theta$ is the scattered angle of the projectile A in the laboratory system. We note that the angle θ^* in **Figure 2** and the angle θ in **Figure 3** are related each other. Once the θ^* is given by the collision problems, the θ is determined according to the prescription stated above. And the converse is also true. When the θ is given in the collision problem, we write down the vector $\boldsymbol{OF} = \boldsymbol{p}_A'$ at first. Then, we

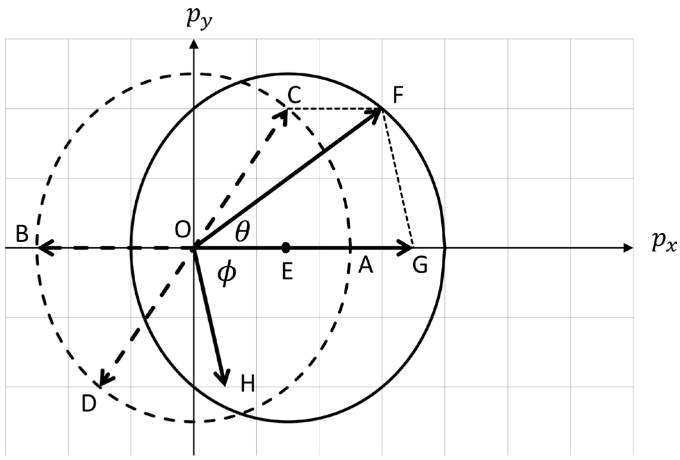

Figure 3. Circle centered at E describes the collision in the laboratory system. The initial condition in **Figure 2** shows that $OE = m_A V = 3$ and $EG = m_B V = 4$. The vectors $\boldsymbol{OF} = \boldsymbol{p}'_A$ and $\boldsymbol{OH} = \boldsymbol{p}'_B$ indicate the momenta of the projectile and the target after the collision in the laboratory system.

draw a broken line from the point F to C. The vector $\boldsymbol{OC} = \boldsymbol{p}'^*_A$ and the angle θ^* are the momentum and the scattered angle of the projectile A in the center-of-mass system.

Next, we determine the point G on the p_x-axis so that $\boldsymbol{EG} = m_B \boldsymbol{V}$. Then the vector $\boldsymbol{FG} = \boldsymbol{OH} = \boldsymbol{p}'_B$ shows the momentum of the target B after the collision. The angle $\angle GOH = \angle FGO = \phi$ is the scattered angle of the target B. The vector relation $\boldsymbol{OG} = \boldsymbol{OE} + \boldsymbol{EG} = \boldsymbol{OF} + \boldsymbol{OH}$ shows the momentum conservation $\boldsymbol{p}_A + \boldsymbol{p}_B = \boldsymbol{p}'_A + \boldsymbol{p}'_B$ before and after the collision.

In contrast with the equations in the previous section, what we need to do are the calculation of the radius of the circle and the lengths OE and EG. They are uniquely determined from the initial condition of the collision. We fix the angle θ or θ^* according to the given collision problems.

4. $\boldsymbol{v}_B = \boldsymbol{0}$ Case

Let us consider the case where the target B is at rest $\boldsymbol{v}_B = \boldsymbol{0}$ before the collision. In that case, the point G definitely lies on the circle centered at E because of $|\boldsymbol{OC}| = |\boldsymbol{EF}| = |\boldsymbol{EG}|$, *i.e.* $p^*_A = p^*_B = m_B V$, which is understood by Equations ((4) and (15)). The circle centered at E is described by Equation (19) and is depicted in **Figure 4**. The vector $\boldsymbol{OG}$ is equal to the momentum $\boldsymbol{p}_A$ of the projectile A before the collision.

The point E lies inside or outside the dashed circle centered at O, according as $m_A < m_B$ and $m_A > m_B$. The corresponding diagrams are shown in **Figure 4** and **Figure 5**. It is evident from these figures that θ and ϕ can be expressed in terms of θ^* by

$$\tan\theta = \frac{EF\sin\theta^*}{OE + EF\cos\theta^*} = \frac{m_B V\sin\theta^*}{m_A V + m_B V\cos\theta^*} = \frac{\sin\theta^*}{\cos\theta^* + \dfrac{m_A}{m_B}}, \tag{23}$$

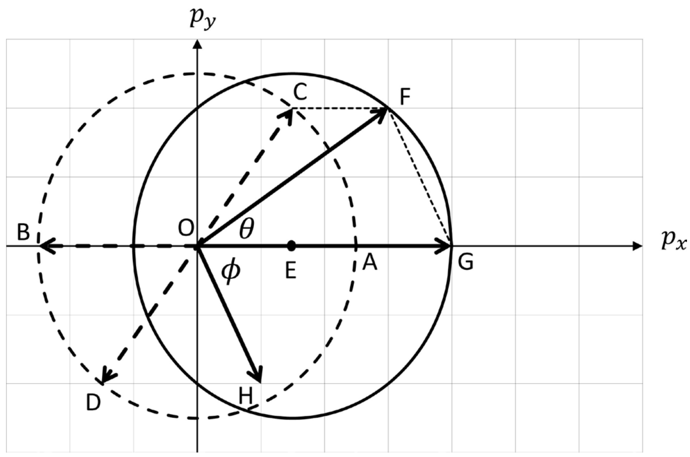

Figure 4. The initial conditions $m_A = 3$, $v_A = \frac{8}{3}$ and $m_B = 5$, $v_B = 0$ show $OC = EF = 5$ and $V = 1$. Then, $OE = m_A V = 3$ and $EG = m_B V = 5$.

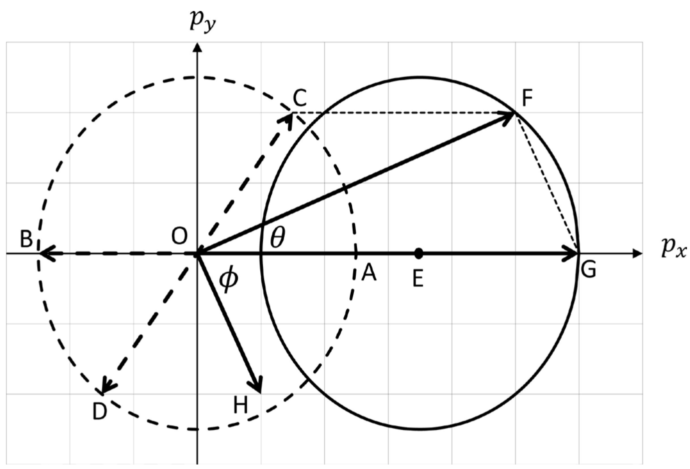

Figure 5. The initial conditions $m_A = 7$, $m_B = 5$, $v_A = \frac{12}{7}$ and $v_B = 0$ show $OC = EF = 5$ and $V = 1$. Then, $OE = m_A V = 7$ and $EG = m_B V = 5$.

$$\tan\phi = \frac{EF\sin\theta^*}{EG - EF\cos\theta^*} = \frac{\sin\theta^*}{1-\cos\theta^*} = \frac{\sin\phi^*}{1+\cos\phi^*}, \tag{24}$$

where we use the relation $\theta^* + \phi^* = \pi$. It is also evident that since the triangle ΔEFG is an isosceles triangle, we obtain $\phi = \frac{\pi - \theta^*}{2}$.

Applying the law of cosine to the triangle ΔOEF: $OF^2 = OE^2 + EF^2 - 2\cdot OE\cdot EF\cos\left(\pi - \theta^*\right)$, we obtain

$$v_A' = \frac{v_A}{m_A + m_B}\sqrt{m_A^2 + m_B^2 + 2m_A m_B\cos\theta^*}. \tag{25}$$

The same application to the triangle ΔEFG gives

$$v_B' = \frac{2m_A v_A}{m_A + m_B}\sin\frac{\theta^*}{2}. \tag{26}$$

If $m_A > m_B$, however, the projectile A can be deflected only through an angle not exceeding $\theta_{\max} = \angle FOA$ from its original direction, as shown in **Figure 6**. This maximum value of $\theta_{\max}$ corresponds to the position F at which OF is a tangent to the circle centered at E. Evidently,

$$\sin\theta_{max} = \frac{EF}{OE} = \frac{EG}{OE} = \frac{m_B V}{m_A V} = \frac{m_B}{m_A}, \quad (27)$$

because $EF = EG$ are both the radius of the circle.

The case $m_A = m_B$ becomes quite simple as shown in **Figure 7**. The circle centered at E is described by Equation (20). The point E lies on the dashed circle centered at O, *i.e.* $OA = OE$. The p_y-axis is a tangent to the circle centered at E. It is evident from the figure that $\theta^* = 2\theta$. Further, Equations ((25) and (26)) become

$$v'_A = v_A \cos\frac{\theta^*}{2}, \quad v'_B = v_A \sin\frac{\theta^*}{2}. \quad (28)$$

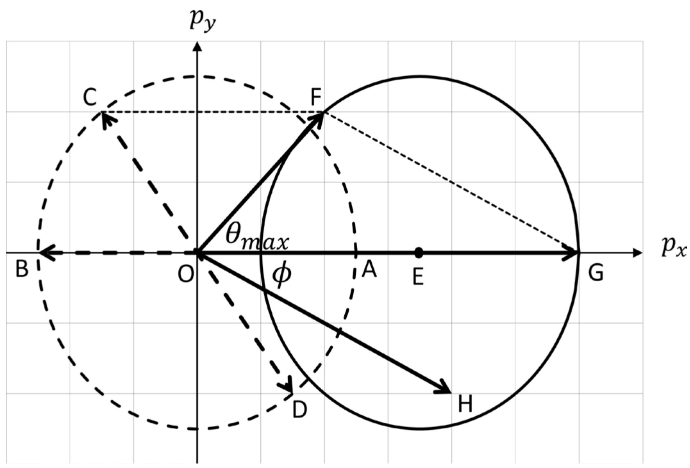

Figure 6. The initial condition is as the same with **Figure 5**. If $m_A > m_B$, the projectile A can be deflected only through an angle not exceeding $\theta_{\max} = \angle FOA$ from its original direction.

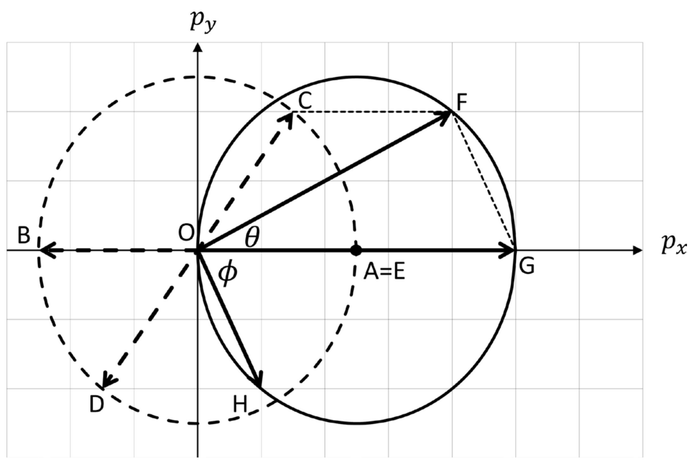

Figure 7. The initial conditions $m_A = m_B = 5$, $v_A = 2$ and $v_B = 0$ show $EF = 5$ and $V = 1$. Then, $m_A V = OE = 5 = m_B V = EG$.

After the collision, the outgoing particles move at right angles to each other, that is $\theta + \phi = \frac{\pi}{2}$.

5. Conclusion

We introduce two circles to analyze the two dimensional elastic collision in momentum space. One circle is for the center-of-mass system and the other is for the laboratory system. The relation between these two systems is clearly understood from these circles. Once we fix the scattered angle of projectile in one system, then we deduce all quantities, such as momenta and scattering angles of both particles in another system. The scattering problem in the special relativistic case is carried out in the next article.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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