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Some Inequalities on T_3 Tree

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Abstract

The article proves several inequalities derived from nodal multiplication on T_3 tree. The proved inequalities are helpful to estimate certain quantities related with the T_3 tree as well as examples of proving an inequality embedded with the floor functions.

Keywords

Inequality, Floor Function, Binary Tree

1. Introduction

The T_3 tree, which first appeared in [1] and was formerly introduced in [2], is a perfect complete binary tree that is considered to be a new tool to study integers. The tree can reveal many new properties of integers such as the symmetric properties discovered in [3] and [4], the genetic property found in [5], and other properties introduced in [6] and [7]. The tree also shows its big potentiality in factorization of big semiprimes, as seen in [8] and [9]. A recent study found several inequalities related with estimation of multiplication on the tree. This article introduces the main results.

2. Preliminaries

This section lists for later sections the necessary preliminaries, which include definitions, notations and lemmas.

2.1. Definitions and Notations

Symbol T_3 is the T_3 tree that was introduced in [1] and [2] and symbol $N_{(k,j)}$ is by default the node at position j on level k of T_3, where $k \ge 0$ and $0 \le j \le 2^k - 1$. Number of the level by default begins at zero and index of the position also by default begins at zero. Symbol $\lfloor x \rfloor$ is the floor function, an integer function of real number x that satisfies inequality $x-1<\lfloor x \rfloor \le x$, or equivalently $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$. Symbol $A \Rightarrow B$ means conclusion B can be derived from condition A.

For convenience in deduction of a formula, comments are inserted by symbols that express their related mathematical foundations. For example, the following deduction

$$A = B$$
$$(L) = C$$
$$(P) \le D$$

means that, lemma (L) supports the step from B to C, and proposition (P) supports the step from C to D.

2.2. Lemmas

Lemma 1. (See in [1]) T_3 tree has the following fundamental properties.

(P1). Every node is an odd integer and every odd integer bigger than 1 must be on the T_3 tree. Odd integer N with $N>1$ lies on level $\lfloor \log_2 N \rfloor - 1$.

(P2). On level k with $k=0,1,\cdots$, there are 2^k nodes starting by $2^{k+1}+1$ and ending by $2^{k+2}-1$, namely, $N_{(k,j)} \in \left[2^{k+1}+1, 2^{k+2}-1\right]$ with $j=0,1,\cdots,2^k-1$.

(P3). $N_{(k,j)}$ is calculated by

$$N_{(k,j)} = 2^{k+1}+1+2j, j=0,1,\cdots,2^k-1$$

(P4). Multiplication of arbitrary two nodes of T_3, say $N_{(m,\alpha)}$ and $N_{(n,\beta)}$, is a third node of T_3. Let $J = 2^m(1+2\beta)+2^n(1+2\alpha)+2\alpha\beta+\alpha+\beta$; the multiplication $N_{(m,\alpha)} \times N_{(n,\beta)}$ is given by

$$N_{(m,\alpha)} \times N_{(n,\beta)} = 2^{m+n+2}+1+2J$$

If $J<2^{m+n+1}$, then $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+1,J)}$ lies on level $m+n+1$ of T_3; whereas, if $J \ge 2^{m+n+1}$, $N_{(m,\alpha)} \times N_{(n,\beta)} = N_{(m+n+2,\chi)}$ with $\chi = J-2^{m+n+1}$ lies on level $m+n+2$ of T_3.

Lemma 2. (See in [10]) Let α and x be a positive real numbers; then it holds

$$\alpha\lfloor x \rfloor - 1 < \lfloor \alpha x \rfloor < \alpha\left(\lfloor x \rfloor + 1\right)$$

Particularly, if α is a positive integer, say $\alpha = n$, then it yields

$$n\lfloor x \rfloor \le \lfloor nx \rfloor \le n\left(\lfloor x \rfloor + 1\right) - 1$$

3. Main Results with Proofs

Proposition 1. For positive integer k and real number $x>0$, it holds

$$0 \ge 2^k \left\lfloor \frac{x}{2^k} \right\rfloor - \lfloor x \rfloor \ge \begin{cases} 1-2^k, 0 \le k \le \lfloor \log_2 x \rfloor \\ -\lfloor x \rfloor, k > \lfloor \log_2 x \rfloor \end{cases} \tag{1}$$

Proof. It can see by Lemma 2 that,

$$2^k\left\lfloor \frac{x}{2^k}\right\rfloor \le \left\lfloor 2^k\cdot\frac{x}{2^k}\right\rfloor = \lfloor x\rfloor$$

and

$$2^k\left\lfloor \frac{x}{2^k}\right\rfloor - \lfloor x\rfloor \ge \left(\left\lfloor 2^k\frac{x}{2^k}\right\rfloor + 1 - 2^k\right) - \lfloor x\rfloor = 1-2^k$$

Meanwhile, when $2^k > x$, or $k > \log_2 x \ge \lfloor \log_2 x\rfloor \ge 0$, $\left\lfloor \frac{x}{2^k}\right\rfloor = 0$; thus

$$2^k\left\lfloor \frac{x}{2^k}\right\rfloor - \lfloor x\rfloor = -\lfloor x\rfloor.$$

Consequently (1) holds.

Proposition 2. Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ be nodes of T_3 with $0\le m\le n$; let

$$J = \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2} - 2^{m+n+1} \tag{2}$$

then when $J < 2^{m+n+1}$

$$2 \le \left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+1}}\right\rfloor \le 3$$

$$\left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+2}}\right\rfloor = 1$$

$$\left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+2+\sigma}}\right\rfloor_{(\sigma\ge 0)} = 0$$

and when $J \ge 2^{m+n+1}$

$$2 \le \left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+2}}\right\rfloor \le 3$$

$$\left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+3}}\right\rfloor = 1$$

$$\left\lfloor \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+3+\sigma}}\right\rfloor_{(\sigma\ge 0)} = 0$$

Proof. By Lemma 1 (P4), it knows, when $J < 2^{m+n+1}$, $N_{(m,\alpha)}\times N_{(n,\beta)}$ lies on level $m+n+1$ of T_3 and thus $2^{m+n+2}+1 \le N_{(m,\alpha)}\times N_{(n,\beta)} \le 2^{m+n+3}-1$; hence it holds

$$2 = \frac{2^{m+n+2}}{2^{m+n+1}} \le \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+1}} \le \frac{2^{m+n+3}-2}{2^{m+n+1}} < 4$$

and

$$1 = \frac{2^{m+n+2}}{2^{m+n+2}} \le \frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{m+n+2}} \le \frac{2^{m+n+3}-2}{2^{m+n+2}} = 2 - \frac{1}{2^{m+n+1}} < 2$$

Thus

$$2 \le \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+1}} \right\rfloor \le 3$$

and

$$\left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \right\rfloor = 1$$

and thus

$$\left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2+\sigma}} \right\rfloor_{(\sigma \ge 1)} = 0$$

Similarly, when $J \ge 2^{m+n+1}$, $N_{(m,\alpha)} \times N_{(n,\beta)}$ lies on level $m+n+2$ of T_3 and $2^{m+n+3} + 1 \le N_{(m,\alpha)} \times N_{(n,\beta)} \le 2^{m+n+4} - 1$ and it holds

$$2 = \frac{2^{m+n+3}}{2^{m+n+2}} < \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \le \frac{2^{m+n+4} - 2}{2^{m+n+2}} = 4 - \frac{1}{2^{m+n+1}} < 4$$
$$\Rightarrow 2 \le \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+2}} \right\rfloor \le 3$$

and

$$1 = \frac{2^{m+n+3}}{2^{m+n+3}} < \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3}} \le \frac{2^{m+n+4} - 2}{2^{m+n+3}} = 2 - \frac{1}{2^{m+n+2}} < 2$$
$$\Rightarrow \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3}} \right\rfloor = 1 \Rightarrow \left\lfloor \frac{N_{(m,\alpha)} \times N_{(n,\beta)} - 1}{2^{m+n+3+\sigma}} \right\rfloor_{(\sigma \ge 1)} = 0$$

Proposition 3. Let $N_{(m,\alpha)}$ be a node of T_3 and n be an integer with $0 \le m \le n$; then it holds

$$-1 < \frac{N_{(m,\alpha)} - 1}{2^{n+1}} - 1 < 1 \tag{3}$$

$$0 < \frac{N_{(m,\alpha)} + 1}{2^{n+1}} \le 2 \tag{4}$$

Thus for arbitrary integer $\sigma \ge 0$

$$-\frac{1}{2^{\sigma}} < \frac{N_{(m,\alpha)} - 1}{2^{n+1+\sigma}} - \frac{1}{2^{\sigma}} < \frac{1}{2^{\sigma}} \tag{5}$$

$$0 < \frac{N_{(m,\alpha)} + 1}{2^{n+1+\sigma}} \le 2^{1-\sigma} \tag{6}$$

Proof. Considering that $2^{m+1} + 1 \le N_{(m,\alpha)} \le 2^{m+2} - 1$ holds for arbitrary $m \ge 0$, it yields

$$-1 + \frac{1}{2^{n-m}} = \frac{2^{m+1}}{2^{n+1}} - 1 \le \frac{N_{(m,\alpha)} - 1}{2^{n+1}} - 1 \le \frac{2^{m+2} - 2}{2^{n+1}} - 1 = \frac{1}{2^{n-m-1}} - \frac{1}{2^{n}} - 1 \tag{7}$$

and

$$0 < \frac{1}{2^{n-m}} + \frac{1}{2^{n}} = \frac{2^{m+1} + 2}{2^{n+1}} \le \frac{N_{(m,\alpha)} + 1}{2^{n+1}} \le \frac{2^{m+2}}{2^{n+1}} = \frac{2}{2^{n-m}} \le 2 \tag{8}$$

Consider in (7)

$$\frac{1}{2^{n-m-1}}-\frac{1}{2^n}-1=\begin{cases}1-\frac{1}{2^n}<1, n=m\\ -\frac{1}{2^n}<0, n=m+1\\ \frac{1}{2^{n-m-1}}-\frac{1}{2^n}-1<0, n>m+1\end{cases}$$

and

$$\frac{1}{2^{n-m}}-1=\begin{cases}0, n=m\\ -1+\frac{1}{2^{n-m}}>-1, n>m\end{cases}$$

it knows (3) and (4) hold and consequently (5) and (6) hold.

Proposition 4. Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ be nodes of T_3 with $0\le m\le n$; then it holds

$$N_{(m,\alpha)}+\frac{N_{(m,\alpha)}-1}{2^{n+1}}\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\le 2N_{(m,\alpha)}-\frac{N_{(m,\alpha)}+1}{2^{n+1}} \tag{9}$$

and thus for arbitrary integer $\sigma\ge 0$ it holds

$$\frac{N_{(m,\alpha)}}{2^{\sigma}}+\frac{N_{(m,\alpha)}-1}{2^{n+1+\sigma}}\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1+\sigma}}\le\frac{N_{(m,\alpha)}}{2^{\sigma-1}}-\frac{N_{(m,\alpha)}+1}{2^{n+1+\sigma}} \tag{10}$$

Consequently, it yields

$$N_{(m,\alpha)}\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\le 2N_{(m,\alpha)}-1 \tag{11}$$

$$\frac{N_{(m,\alpha)}-1}{2}-1\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-1 \tag{12}$$

and

$$\frac{N_{(m,\alpha)}}{2^2}-2<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le\frac{N_{(m,\alpha)}-1}{2} \tag{13}$$

$$\frac{N_{(m,\alpha)}-1}{2}-2\le 2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1 \tag{13*}$$

Proof. The condition that $N_{(n,\beta)}$ is a node of T_3 leads to

$$2^{n+1}+1\le N_{(n,\beta)}\le 2^{n+2}-1$$

Then direct calculation shows

$$\begin{aligned}&\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}-N_{(m,\alpha)}-\frac{N_{(m,\alpha)}-1}{2^{n+1}}\\&=\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1-2^{n+1}N_{(m,\alpha)}-N_{(m,\alpha)}+1}{2^{n+1}}\\&=\frac{N_{(m,\alpha)}\times\left(N_{(n,\beta)}-\left(2^{n+1}+1\right)\right)}{2^{n+1}}\ge 0\end{aligned}$$

and

$$\begin{aligned}&\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}-2N_{(m,\alpha)}+\frac{N_{(m,\alpha)}+1}{2^{n+1}}\\&=\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1-2^{n+2}N_{(m,\alpha)}+N_{(m,\alpha)}+1}{2^{n+1}}\\&=\frac{N_{(m,\alpha)}\times\left(N_{(n,\beta)}-\left(2^{n+2}-1\right)\right)}{2^{n+1}}\le 0\end{aligned}$$

Hence (9) holds.

Multiplying each item in (9) by $\frac{1}{2^{\sigma}}$ for integer $\sigma\ge 1$ immediately yields (10).

By definition of the floor function, it holds

$$\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}$$

By the Inequalities (3), (4) and (9) it yields

$$\begin{aligned}&N_{(m,\alpha)}+\underline{\underline{\frac{N_{(m,\alpha)}-1}{2^{n+1}}}}-1\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\le 2N_{(m,\alpha)}-1\\&\Rightarrow N_{(m,\alpha)}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\le 2N_{(m,\alpha)}-1\\&\Rightarrow N_{(m,\alpha)}\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\le 2N_{(m,\alpha)}-1\end{aligned}$$

which says (11) holds.

Likewise, by definition of the floor function and referring to the Inequalities (5), (6) and (10), it yields

$$\begin{aligned}&\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\\&\Rightarrow\frac{N_{(m,\alpha)}}{2}+\frac{N_{(m,\alpha)}-1}{2^{n+2}}-1\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-\frac{N_{(m,\alpha)}+1}{2^{n+2}}\\&\Rightarrow\frac{N_{(m,\alpha)}}{2}+\frac{N_{(m,\alpha)}-1}{2^{n+2}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-1\\&\Rightarrow\frac{N_{(m,\alpha)}-1}{2}+\underline{\underline{\frac{N_{(m,\alpha)}-1}{2^{n+2}}-\frac{1}{2}}}<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-1\\&\Rightarrow\frac{N_{(m,\alpha)}-1}{2}-\frac{1}{2}<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-1\\&\Rightarrow\frac{N_{(m,\alpha)}-1}{2}-1\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le N_{(m,\alpha)}-1\end{aligned}\tag{14}$$

which is the (12).

Similarly, the Inequalities (10) and the definition of the floor function lead to

$$\frac{N_{(m,\alpha)}}{2^2}+\frac{N_{(m,\alpha)}-1}{2^{n+3}}\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\le\frac{N_{(m,\alpha)}}{2}-\frac{N_{(m,\alpha)}+1}{2^{n+3}}$$

and

$$\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}$$

Then referring to the Inequalities (5) and (6), it immediately results in

$$\begin{aligned}
&\frac{N_{(m,\alpha)}}{2^2}+\frac{N_{(m,\alpha)}-1}{2^{n+3}}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le\frac{N_{(m,\alpha)}}{2}-\frac{N_{(m,\alpha)}+1}{2^{n+3}}\\
&\Rightarrow\frac{N_{(m,\alpha)}}{2^2}+\underline{\underline{\frac{N_{(m,\alpha)}-1}{2^{n+3}}-\frac{1}{2^2}}}-\frac{3}{4}<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor<\frac{N_{(m,\alpha)}}{2}\\
&\Rightarrow\frac{N_{(m,\alpha)}}{2^2}-\frac{1}{4}-\frac{3}{4}<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor<\frac{N_{(m,\alpha)}-1}{2}+\frac{1}{2}\\
&\Rightarrow\frac{N_{(m,\alpha)}}{2^2}-1<\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le\frac{N_{(m,\alpha)}-1}{2}\\
&\Rightarrow\frac{N_{(m,\alpha)}}{2}-2<2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1\\
&\Rightarrow\frac{N_{(m,\alpha)}-1}{2}-\frac{3}{2}<2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1 \qquad (15)\\
&\Rightarrow\frac{N_{(m,\alpha)}-1}{2}-1\le2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1
\end{aligned}$$

which is just the (13).

Proposition 5. Let $N_{(m,\alpha)}$ and $N_{(n,\beta)}$ be nodes of T_3 with $0\le m\le n$; then it holds for integer $0\le s\le m$

$$N_{(m,\alpha)}-2^{s+2}+1\le2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le2N_{(m,\alpha)}-1 \qquad (16)$$

and

$$\frac{N_{(m,\alpha)}-1}{2}-2^{s+2}\le2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1 \qquad (17)$$

Proof. By Lemma 2 and Proposition 1, it holds when $0\le s\le m$

$$\begin{aligned}
&2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\\
&(P1)\ge\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n}}\right\rfloor+1-2^{s+2}-\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor\\
&(L2)\ge\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor+1-2^{s+2}
\end{aligned}$$

and

$$2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor$$

$$(L2)\le\left\lfloor 2^{s+2}\times\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+s+2}}\right\rfloor-\left(\left\lfloor 2\times\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor+1-2\right)$$

$$=\left\lfloor 2\times\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor-\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor+1$$

$$(L2)\le 2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor-1-\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor+1$$

$$=\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor$$

That is

$$\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor+1-2^{s+2}$$

$$\le 2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+1}}\right\rfloor$$

By (11) it holds

$$N_{(m,\alpha)}+1-2^{s+2}\le 2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor\le 2N_{(m,\alpha)}-1$$

which is just the (16).

Similarly it holds

$$\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor+1-2^{s+2}$$

$$\le 2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+2}}\right\rfloor$$

and by (12) it yields

$$\frac{N_{(m,\alpha)}-1}{2}-2^{s+2}\le 2^{s+2}\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3+s}}\right\rfloor-2\left\lfloor\frac{N_{(m,\alpha)}\times N_{(n,\beta)}-1}{2^{n+3}}\right\rfloor\le N_{(m,\alpha)}-1$$

4. Conclusion

The T_3 tree is emerging its value in studying integers. A lot of equations and inequalities will be research objectives. Since most of the inequalities on the T_3 tree are in the form of floor functions, their proofs are often skillful. The inequalities proved in this article are not only quite useful for knowing the T_3 tree, but also excellent samples for proving inequalities with the floor functions. Hope it helpful to the readers of interests.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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