

Finslerian Ricci Deformation and Conformal Metrics

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Abstract

In this paper, a new Ricci flow is canonically introduced in Finsler Geometry and, under the variance of Finsler-Ehresmann form, conformal changes of Finsler metrics are studied. Some existence conditions of this Finslerian Ricci flow on a compact manifold which preserves the conformal class of the initial metric are obtained as an application.

Keywords

Ehresmann Connection, Ricci Flow, Trace-Free Ricci Tensor, Conformal Change of Finsler-Ehresmann Form

1. Introduction

The Ricci flow is a very powerful tool in studying of the geometry of manifolds and has many applications in Mathematics and Physics. In Finsler Geometry, the problems on Ricci flow are very interesting. In 2017, to study deformation of Cartan curvature, Bidabad, Shahi and Ahmadi considered the Akbar-Zadeh's Ricci curvature and introduced certain Ricci flow for Finsler n -manifolds [1].

In this paper, we use the pulled-back bundle approach [2] to introduce a Finslerian horizontal Ricci flow, called Finslerian Ricci deformation. This approach is natural and is very important. The problem of construction of the Finslerian Ricci deformation contains a number of new conceptual. Let (M, F) be a Finsler manifold of flag scalar curvature. Then we consider a Finslerian Ricci tensor defined by $\mathbf{Ric}_F^H(\xi, X) := \sum_{a=1}^n \mathbf{R}(\xi, e_a, X, \hat{e}_a)$. In this definition, \mathbf{Ric}_F^H denotes again a Ricci tensor associated with F and a Finsler-Ehresmann connection \mathcal{H} , ξ is a section of the vector bundle π^*TM , X is a section of the tangent bundle $\overset{\circ}{T}M$ of $\overset{\circ}{T}M := TM \setminus \{0\}$, \mathbf{R} is the hh -curvature of Chern

connection, $\{e_a\}_{a=1,\dots,n}$ is the special g -orthonormal basis section for π^*TM and $\{\hat{e}_a\}_{a=1,\dots,n}$ is the orthonormal basis section for \mathcal{H} . Let (M, F_t) be an n -dimensional Finsler manifold and $\{g_t\}_{t \in [0, \tau)}$ (τ being a finite parameter) a family of fundamental tensors of $\{F_t\}_{t \in [0, \tau)}$. We consider the following Finslerian Ricci deformation:

$$\frac{\partial}{\partial t} g_t = -2\mathbf{Ric}_{F_t}^H, \quad g_t|_0 = g, \quad \forall t \in [0, \tau). \quad (1)$$

The existence and uniqueness for solution of the Equation (1) are known in special cases, particularly in Riemannian spaces and Berwald spaces [3]. This Finslerian Ricci deformation generalizes the classical Riemannian one.

A Finsler metric \tilde{F} on M is said to be conformally equivalent to F , if there exists a C^∞ function u on M such that $\tilde{F} = e^u F$. In this paper, $[F]$ denotes the conformal class of F . We prove the following results.

Theorem 1. Let (M, F) be an n -dimensional compact Finsler manifold, F_t a Ricci deformation of F for $t \in [0, \tau)$ and g_t the fundamental tensor of F_t . If $[F_t] = [F]$ for all $t \in [0, \tau)$ then there exists a family $\{u_t\}_{t \in [0, \tau)}$ of C^∞ functions on M satisfying the following equation.

$$\begin{aligned} \frac{\partial u_t}{\partial t} = & -\frac{1}{n} e^{-2u_t} \left[\mathbf{Scal}_F^H + 2(n-1)\Delta^H u_t - (n-1)(n-2)\|\nabla u_t\|_g^2 \right] \\ & - \frac{2-n}{n} e^{-2u_t} \sum_{a=1}^n \left[g(\nabla u_t, \Theta(\hat{e}_a, \hat{e}_b)) - g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u_t)), e_a) \right] \\ & - \frac{1}{n} e^{-2u_t} \sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a) \right] \\ & - \frac{1}{n} e^{-2u_t} \sum_{a,b=1}^n \left[g((\nabla_{\hat{e}_b} \Theta)(\hat{e}_a, \hat{e}_b), e_a) - g((\nabla_{\hat{e}_a} \Theta)(\hat{e}_b, \hat{e}_b), e_a) \right] \end{aligned} \quad (2)$$

with $u_t|_{t=0} = 0$, where \mathbf{Scal}_F^H is the horizontal scalar curvature, ∇ is the Chern connection, \mathbf{h} denotes a horizontal lift of a section of π^*TM , Δ^H is the horizontal Laplacian, ∇ is the gradient and Θ is the $(0, 2; 1)$ -tensor on M measuring the variation of ∇ during the conformal change of F .

Corollary 1. Assume $n \geq 2$. Then the Equation (2) in Theorem 1 has a unique solution on a parameterized-interval $[0, s)$ for some $s \leq \tau$.

Finally, using the trace-free horizontal Ricci tensor we prove the following

Theorem 2. Let (M, F) be an n -dimensional compact Finsler-Einstein manifold, g its fundamental tensor and u_t the unique solution of Equation (2) on $[0, s) \times M$ for some $s \leq \tau$, where $[0, \tau)$ is the maximal parameterized-interval on which (1) has a solution. Assume that the tensor T_t , defined for a $t \in [0, s)$ and in local coordinate by

$$\begin{aligned} T_{cd}(t) = & e^{-2u_t} \left\{ -(n-2) \left[H_{u_t}(e_c, e_d^H) - du_t(e_c) du_t(e_d) \right] \right. \\ & + \sum_{a=1}^n \left[(2-n) g(\nabla u_t, \Theta(\hat{e}_a, \hat{e}_a)) - g(\Theta(\mathbf{h}(\nabla u_t), \hat{e}_a), e_a) \right] \delta_{cd} \\ & \left. + \sum_{a=1}^n \left[-g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u_t)), e_a) + g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u_t)), e_a) \right] \delta_{cd} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{a,b=1}^n \left[g\left(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a\right) - g\left(\Theta(\hat{e}_a, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a\right) \right] \delta_{cd} \\
 & + \sum_{a,b=1}^n \left[g\left((\nabla_{\hat{e}_b} \Theta)(\hat{e}_a, \hat{e}_b), e_a\right) - g\left((\nabla_{\hat{e}_a} \Theta)(\hat{e}_b, \hat{e}_b), e_a\right) \right] \delta_{cd},
 \end{aligned} \tag{3}$$

is conformally equivalent to g . Then there exists a unique Ricci deformation $\{F_t\}_{t \in [0,s]}$ of F such that $[F_t] = [F]$.

The rest of this paper is organised as follows. In Section 2, we give some basic notions on Finsler manifolds. In Section 3, we prove the main results given above.

2. Preliminaries

In order to deal with the Finslerian Ricci deformation, it is preferable to use a global definition of Chern connection. We adopt the notations given in [2] and [4].

Let $\pi : TM \rightarrow M$ be the tangent bundle of a connected C^∞ manifold of dimension $n \geq 3$. We denote by $\mathring{TM} = TM \setminus \{0\}$ the slit tangent bundle of M . We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with $(x^i)_{i=1, \dots, n}$ a local coordinate on U . By setting $y = y^i \frac{\partial}{\partial x^i}$ for every $y \in \pi^{-1}(U)$, we introduce a local coordinate $(x^i, y^i)_{i=1, \dots, n}$ on $\pi^{-1}(U) \subset TM$. Throughout this paper, we use Einstein summation convention for the expressions with indices when an index appears twice as a subscript as well as a superscript in a term.

Definition 2.1. A function $F : TM \rightarrow [0, \infty)$ is called a Finsler metric on M if:

- 1) F is C^∞ on the entire slit tangent bundle \mathring{TM} ,
- 2) F is positively 1-homogeneous on the fibers of TM , that is $\forall c > 0, F(x, cy) = cF(x, y)$,
- 3) the Hessian matrix $(g_{ij}(x, y))_{1 \leq i, j \leq n}$ with elements

$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \tag{4}$$

is positive definite at every point (x, y) of \mathring{TM} .

Given a manifold M and a Finsler metric F on TM , the pair (M, F) is called a *Finsler manifold*.

Remark 2.1. $F(x, y) \neq 0$ for all $x \in M$ and for every $y \in T_x M \setminus \{0\}$.

The pulled-back bundle $\pi^* TM$ is a vector bundle over the slit tangent bundle \mathring{TM} . The fiber at a point $(x, y) \in \mathring{TM}$ is defined by

$$\pi^* TM \Big|_{(x,y)} := \left\{ (x, y, v) \mid v \in T_{\pi(x,y)} M \right\} \Big|_{(x,y)} \cong T_x M. \tag{5}$$

By the objects (4), the vector bundle $\pi^* TM$ admits a Riemannian metric.

$g := g_{ij} dx^i \otimes dx^j$ called fundamental tensor. Likewise, there is the Finslerian Cartan tensor

$$A = A_{ijk} dx^i \otimes dx^j \otimes dx^k \quad (6)$$

where $A_{ijk} := \frac{F}{2} \frac{\partial g_{ij}}{\partial y^k}$. A is a symmetric section of $\otimes^3 \pi^* T^*M$. Note that, $\frac{\partial}{\partial x^i}$ and dx^i are respectively regarded as basis sections of $\pi^* TM$ and $\pi^* T^*M$ (see [2]).

2.1. Finsler-Ehresmann Connection and Chern Connection

For the differential mapping π_* of the submersion $\pi: \overset{\circ}{T}M \rightarrow M$, the vertical subbundle \mathcal{V} of $\overset{\circ}{T}M$ is defined by $\mathcal{V} := \ker(\pi_*)$ and it is locally spanned by the set $\left\{ F \frac{\partial}{\partial y^i}, 1 \leq i \leq n \right\}$, on each $\pi^{-1}(U) \subset TM$. Then it induces the short exact sequence

$$0 \rightarrow \mathcal{V} \xrightarrow{i} \overset{\circ}{T}M \xrightarrow{\pi_*} \pi^* TM \rightarrow 0. \quad (7)$$

A horizontal subspace \mathcal{H} of $\overset{\circ}{T}M$ is by definition any complementary to \mathcal{V} . The bundles \mathcal{H} and \mathcal{V} give a smooth splitting [5]

$$\overset{\circ}{T}M = \mathcal{H} \oplus \mathcal{V}. \quad (8)$$

The vertical bundle \mathcal{V} is uniquely determined but the horizontal bundle \mathcal{H} is not canonically determined. An Ehresmann connection is a selection of horizontal subspace \mathcal{H} of $\overset{\circ}{T}M$.

In this paper, we consider the choice of Ehresmann connection which arises from the Finsler metric F and it is called Finsler-Ehresmann connection [6], constructed as follows. As explained in [7], all Finsler structure F on M induces a vector field on $\overset{\circ}{T}M$ in the form

$$G(x, y) = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y) \frac{\partial}{\partial y^i}$$

where $x = (x^i) \in M, y = y^i \frac{\partial}{\partial x^i} \in \overset{\circ}{T}M$ and the elements

$$G^i(x, y) := \frac{1}{4} g^{il} \left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right) y^j y^k$$

are y -homogeneous of degree two. The vector field G is called *spray* on M and the $G^i, i = 1, \dots, n$ are called *spray coefficients* of G .

Consider the functions

$$N_j^i(x, y) := \frac{\partial G^i}{\partial y^j}(x, y), 1 \leq i, j \leq n$$

one defines a $\pi^* TM$ -valued C^∞ form on $\overset{\circ}{T}M$ by

$$\theta = \frac{\partial}{\partial x^i} \otimes \frac{1}{F} (dy^i + N_j^i dx^j). \quad (9)$$

This $\pi^* TM$ -valued C^∞ form θ is globally well defined on $\overset{\circ}{T}M$ [4].

From the form θ , called Finsler-Ehresmann form, defined in (9), one defines

a Finsler-Ehresmann connection as follows.

Definition 2.2. A Finsler-Ehresmann connection associated with the submersion.

$\pi : \overset{\circ}{TM} \rightarrow M$ is the subbundle \mathcal{H} of $T\overset{\circ}{TM}$ given by $\mathcal{H} := \ker \theta$, where θ is the bundle morphism from $T\overset{\circ}{TM}$ to π^*TM defined in (9), and which is complementary to the vertical subbundle $\mathcal{V} \subset T\overset{\circ}{TM}$.

Now we define horizontal lift and vertical lift of a section of π^*TM as follows.

Definition 2.3. Let $\pi : \overset{\circ}{TM} \rightarrow M$ be the restricted projection.

1) The form $\theta : T\overset{\circ}{TM} \rightarrow \pi^*TM$ induces a linear map

$$\theta|_{(x,y)} : T_{(x,y)}\overset{\circ}{TM} \rightarrow T_xM, \text{ for each point } (x,y) \in \overset{\circ}{TM}; \tag{10}$$

where $x = \pi(x,y)$.

The vertical lift of a section ξ of π^*TM is a unique section $\mathbf{v}(\xi)$ of $T\overset{\circ}{TM}$ such that for every $(x,y) \in \overset{\circ}{TM}$

$$\pi_*(\mathbf{v}(\xi))|_{(x,y)} = 0_{(x,y)} \text{ and } \theta(\mathbf{v}(\xi))|_{(x,y)} = \xi_{(x,y)}. \tag{11}$$

2) The differential projection $\pi_* : T\overset{\circ}{TM} \rightarrow \pi^*TM$ induces a linear map

$$\pi_*|_{(x,y)} : T_{(x,y)}\overset{\circ}{TM} \rightarrow T_xM, \text{ for each point } (x,y) \in \overset{\circ}{TM}; \tag{12}$$

where $x = \pi(x,y)$.

The horizontal lift of a section ξ of π^*TM is a unique section $\mathbf{h}(\xi)$ of $T\overset{\circ}{TM}$ such that for every $(x,y) \in \overset{\circ}{TM}$

$$\pi_*(\mathbf{h}(\xi))|_{(x,y)} = \xi_{(x,y)} \text{ and } \theta(\mathbf{h}(\xi))|_{(x,y)} = 0_{(x,y)}. \tag{13}$$

Remark 2.2. The vector bundle π^*TM can be naturally identified with the horizontal subbundle \mathcal{H} of $T\overset{\circ}{TM}$ or with the vertical bundle \mathcal{V} . Thus any section ξ of π^*TM is considered as a section of \mathcal{H} or a section of \mathcal{V} . In fact

$$\xi = \xi^i \frac{\partial}{\partial x^i} \in \Gamma(\pi^*TM) \Leftrightarrow \mathbf{h}(\xi) = \xi^i \frac{\delta}{\delta x^i} \in \Gamma(\mathcal{H}),$$

and

$$\xi = \xi^i \frac{\partial}{\partial x^i} \in \Gamma(\pi^*TM) \Leftrightarrow \mathbf{v}(\xi) = \xi^i F \frac{\partial}{\partial y^i} \in \Gamma(\mathcal{V}),$$

where

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j} = h\left(\frac{\partial}{\partial x^i}\right) \right\}_{i=1,\dots,n} \text{ and } \left\{ F \frac{\partial}{\partial y^i} := v\left(\frac{\partial}{\partial x^i}\right) \right\}_{i=1,\dots,n}$$

are respectively horizontal and vertical lifts of the natural basis $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1,\dots,n}$ for π^*TM .

The following theorem defines the Chern connection on the bundle π^*TM .

Theorem 3. [4] Let (M, F) be a Finsler manifold, g the fundamental tensor associated with F and π_* the differential mapping of the submersion

$\pi : \overset{\circ}{TM} \rightarrow M$. There exist a unique linear connection ∇ on the pulled-back tangent bundle π^*TM such that, for all

$$X, Y \in \Gamma\left(\overset{\circ}{TM}\right) \text{ and } \xi, \eta \in \Gamma\left(\pi^*TM\right),$$

one has the following properties:

- 1) Symmetry: $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_* [X, Y]$,
- 2) Almost g -compatibility:

$$X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) + 2\mathcal{A}(\theta(X), \xi, \eta),$$

where \mathcal{A} is the Cartan tensor defined in (6) and θ is the Finsler-Ehresmann form defined in (9).

2.2. On Finslerian Curvatures of Chern Connection

Let \mathcal{E} be a vector bundle over $\overset{\circ}{TM}$. Then, one denotes by $\Gamma^p(\mathcal{E})$ the $C^\infty(TM)$ -module of differentiable sections of \mathcal{E}^p , where

$$\mathcal{E}^p = \underbrace{\mathcal{E} \oplus \dots \oplus \mathcal{E}}_{p\text{-times}}$$

denotes the fibered product of \mathcal{E} . By convention $\Gamma^0(\mathcal{E}) = C^\infty(TM)$. The tensors that will be considered are defined as follows:

Definition 2.4. Let (M, F) be a Finsler manifold. A tensor field T of type $(p_1, p_2; q)$ on (M, F) is a mapping

$$T : \underbrace{\pi^*TM \oplus \dots \oplus \pi^*TM}_{p_1\text{-times}} \oplus \underbrace{\overset{\circ}{TM} \oplus \dots \oplus \overset{\circ}{TM}}_{p_2\text{-times}} \rightarrow \otimes^q \pi^*TM, \quad p_1, p_2 \text{ and } q \in \mathbb{N}$$

which is $C^\infty(TM)$ -linear in each arguments.

The full curvature ϕ , of Chern connection ∇ , is the $(1, 2; 1)$ -tensor defined by

$$\phi(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X, Y]}\xi$$

where $X, Y \in \Gamma\left(\overset{\circ}{TM}\right)$ and $\xi \in \Gamma(\pi^*TM)$. Using the decomposition (8), we have

$$\nabla_X = \nabla_{X^H} + \nabla_{X^V}, \tag{14}$$

where $X = X^H + X^V$ with $X^H \in \Gamma(\mathcal{H})$ and $X^V \in \Gamma(\mathcal{V})$.

The full curvature ϕ can be written as

$$\phi(X, Y)\xi = \phi^{HH}(X, Y)\xi + \phi^{HV}(X, Y)\xi + \phi^{VH}(X, Y)\xi + \phi^{VV}(X, Y)\xi$$

where

$$\phi^{HH}(X, Y)\xi = \phi(X^H, Y^H)\xi = \nabla_{X^H} \nabla_{Y^H} \xi - \nabla_{Y^H} \nabla_{X^H} \xi - \nabla_{[X^H, Y^H]}\xi$$

$$\phi^{HV}(X, Y)\xi = \phi(X^H, Y^V)\xi = \nabla_{X^H} \nabla_{Y^V} \xi - \nabla_{Y^V} \nabla_{X^H} \xi - \nabla_{[X^H, Y^V]}\xi$$

$$\phi^{VH}(X, Y)\xi = \phi(X^V, Y^H)\xi = \nabla_{X^V} \nabla_{Y^H} \xi - \nabla_{Y^H} \nabla_{X^V} \xi - \nabla_{[X^V, Y^H]}\xi$$

$$\phi^{VV}(X, Y)\xi = \phi(X^V, Y^V)\xi = \nabla_{X^V}\nabla_{Y^V}\xi - \nabla_{Y^V}\nabla_{X^V}\xi - \nabla_{[X^V, Y^V]}\xi.$$

Remark 2.3.

1) As in the Riemannian case, one can define a $(2, 2; 0)$ version of ϕ as follows:

$$\begin{aligned} \Phi(\xi, \eta, X, Y) &= g(\phi(X, Y)\xi, \eta) \\ &= g(R(X, Y)\xi, \eta) + g(P(X, Y)\xi, \eta) \\ &= \mathbf{R}(\xi, \eta, X, Y) + \mathbf{P}(\xi, \eta, X, Y) \end{aligned} \tag{15}$$

where \mathbf{R} and \mathbf{P} are respectively the hh - and hv -curvature tensor of the Chern connection. One has

$$\mathbf{R}(\xi, \eta, X, Y) = \Phi(\xi, \eta, X^H, Y^H) \tag{16}$$

and

$$\mathbf{P}(\xi, \eta, X, Y) = \Phi(\xi, \eta, X^H, Y^V) + \Phi(\xi, \eta, X^V, Y^H). \tag{17}$$

2) The hh -curvature tensor \mathbf{R} is a generalization of the usual Riemannian curvature.

3) The hv -curvature tensor \mathbf{P} is a Finsler non Riemannian curvature.

Definition 2.5. Let (M, F) be a Finsler manifold, R the horizontal part of the full curvature tensor associated with the Chern connection. We define

1) the horizontal Ricci tensor \mathbf{Ric}_F^H of (M, F) by

$$\begin{aligned} \mathbf{Ric}_F^H : \pi^*TM \oplus \overset{\circ}{T}TM &\rightarrow C^\infty(TM) \\ (\xi, X) &\mapsto \text{trace}_g(\eta \mapsto R(X, \mathbf{h}(\eta))\xi), \end{aligned} \tag{18}$$

for every $\eta \in \Gamma(\pi^*TM)$. In g -orthonormal basis sections $\{e_a\}_{a=1, \dots, n}$ of π^*TM , we have

$$\mathbf{Ric}_F^H(\xi, X) := \sum_{a=1}^n \mathbf{R}(\xi, e_a, X, \hat{e}_a). \tag{19}$$

2) the horizontal scalar curvature \mathbf{Scal}_F^H of (M, F) is the trace of the horizontal Ricci tensor. \mathbf{Scal}_F^H is a function on $\overset{\circ}{T}M$. In g -orthonormal basis sections $\{e_a\}_{a=1, \dots, n}$ of π^*TM ,

$$\mathbf{Scal}_F^H := \sum_{a=1}^n \mathbf{Ric}_F^H(e_a, \hat{e}_a) = \sum_{a,b=1}^n \mathbf{R}(e_a, e_b, \hat{e}_a, \hat{e}_b). \tag{20}$$

3) Let (M, F) be a Finsler manifold and g its fundamental tensor. Consider a flag $\mathcal{P} \subset T_xM$, that is a 2-dimensional subspace of T_xM , a flag-pole $0 \neq y \in T_xM$ and a transverse edge ξ . A flag curvature is defined by

$$\mathbf{K}(x, y, \mathcal{P}) = \mathbf{K}(x, y, \xi) = \frac{\mathbf{R}(y, \xi, X, Y)}{g(y, y)g(\xi, \xi) - (g(y, \xi))^2} \tag{21}$$

where $\xi \in \mathcal{P}$ is a noncolinear to the vector y , with X and Y such that $\xi = \pi_*X$ and $y = \pi_*Y$.

Remark 2.4. If \mathbf{K} is independent of the transverse edge ξ , then the Finsler

manifold (M, F) is called of scalar flag curvature. Denote this scalar by $\tau = \tau(x, y)$. When $\tau(x, y)$ has no dependence on either x nor y , (M, F) is said to be of constant scalar curvature.

Now, we define the trace-free horizontal Ricci tensor and an Finsler-Einstein metric as follows.

Definition 2.6.

1) The trace-free horizontal Ricci tensor of an n -dimensional Finsler manifold (M, F) is a $(1,1;0)$ -tensor on (M, F) given by

$$B_F^H = Ric_F^H - \frac{1}{n} Scal_F^H \underline{g}, \quad (22)$$

where $\underline{g} := \pi^* g$ that is the pullback of g by the submersion $\pi: \overset{\circ}{TM} \rightarrow M$; and for every $\xi \in \Gamma(\pi^* TM)$ and for any $X \in \Gamma(\overset{\circ}{TM})$, $\underline{g}(\xi, X) = g(\xi, \pi_* X)$, Ric_F^H is the horizontal Ricci tensor and $Scal_F^H$ is the horizontal scalar curvature of (M, F) .

2) An n -dimensional Finsler manifold (M, F) is horizontally Einstein if the trace-free horizontal Ricci tensor associated with F vanishes, that is

$$Ric_F^H = \frac{1}{n} Scal_F^H \underline{g}.$$

In this case, $Scal_F^H$ is a function on M for $n \geq 3$.

2.3. Horizontal Differential Operators on a Finsler Manifold

In this paragraph, we give fundamental horizontal differential operators on (M, F) .

Remark 2.5. A differential operator O of order $2m$ defined on a differentiable manifold M is written as

$$O(u) = f(x, u, \nabla u, \nabla^{2m} u)$$

where $u \in C^{2m}(M)$ and f is assumed to be a differentiable function of its arguments.

Definition 2.7.

1) Let $\tau: \pi^* TM \rightarrow TM$ be the canonical mapping defined by $\tau(x, y, v) = v$. For a smooth function u on M , the gradient of u , noted by ∇u , is the section of $\pi^* TM$, given by

$$g_{(x,y)}(\nabla u_{(x,y)}, \xi_{(x,y)}) = du_{\pi(x,y)}(\tau \xi) \quad (23)$$

for any $\xi \in \Gamma(\pi^* TM)$ and for every $(x, y) \in \overset{\circ}{TM}$. Locally, one has

$$\nabla u_{(x,y)} = g^{ij}(x, y) \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j}. \quad (24)$$

2) For a C^∞ section $\xi \in \Gamma(\pi^* TM)$, we define the horizontal divergence by

$$div^H \xi = trace_g(\eta \mapsto \nabla_{h(\eta)} \xi), \quad (25)$$

where g is the fundamental tensor associated with F and ∇ is the Chern

connection.

Remark 2.6. In the local basis sections $\left\{ \frac{\partial}{\partial x^i} \right\}_{i=1, \dots, n}$ of the bundle π^*TM , we have:

$$\operatorname{div}^H \xi = g^{ij} g \left(\nabla_{\frac{\delta}{\delta x^i}} \xi, \frac{\partial}{\partial x^j} \right), \frac{\delta}{\delta x^i} := \mathbf{h} \left(\frac{\partial}{\partial x^i} \right). \tag{26}$$

Definition 2.8.

1) Let (M, F) be a C^∞ Finsler manifold and ∇ the Chern connection on the pulled-back bundle π^*TM . The horizontal Hessian of a C^∞ function u on M is the map

$$H_u : \Gamma(\pi^*TM) \times \Gamma(T\dot{TM}) \rightarrow C^\infty(TM)$$

such that

$$H_u(\xi, X) = g(\xi, \nabla_X(\nabla u)), \quad \forall \xi \in \pi^*TM, \forall X \in T\dot{TM}. \tag{27}$$

2) Let (M, F) be a C^∞ Finsler manifold and u a C^∞ function on M . The horizontal Laplacian of u are respectively defined by the following relation.

$$\Delta^H u = -\operatorname{div}^H \nabla u. \tag{28}$$

Lemma 4. Let $u \in C^\infty(M)$. The horizontal Laplacian $\Delta^H u$ of u can be given in term of the horizontal Hessian of u by

$$\Delta^H u = -\operatorname{trace}_g((\xi, \eta) \mapsto H_u(\xi, \mathbf{h}(\eta))), \quad \xi, \eta \in \Gamma(\pi^*TM). \tag{29}$$

Furthermore, in g -orthonormale basis sections $\{e_a\}_{a=1, \dots, n}$, one has

$$\Delta^H u = -\sum_{a=1}^n H_u(e_a, \hat{e}_a), \quad \text{with } \hat{e}_a := \mathbf{h}(e_a). \tag{30}$$

Proof. By definition of horizontal Laplacian.

3. Finslerian Ricci Deformation and Conformal Metrics

In this section we prove the main results.

Lemma 5. Let (M, F) be an n -dimensional Finsler manifold. If $\tilde{F} = e^u F$ is Finsler metric conformal to F then the horizontal Ricci tensors $\widetilde{\mathbf{Ric}}^H_{\tilde{F}}$ and \mathbf{Ric}^H_F associated with \tilde{F} and F respectively are conformally related by the equation:

$$\begin{aligned} & \widetilde{\mathbf{Ric}}^H_{\tilde{F}}(\xi, X) \\ &= \left[\mathbf{Ric}^H_F + \left(\Delta^H u - (n-2) \|\nabla u\|_g^2 \right) g - (n-2)(H_u - du \circ du) \right](\xi, X) \\ & \quad + (2-n)g(\nabla u, \Theta(X, \mathbf{h}(\xi))) - g(\Theta(\mathbf{h}(\nabla u), \mathbf{h}(\xi)), \pi_* X) \\ & \quad - g(\Theta(X, \mathbf{h}(\nabla u)), \xi) + \sum_{a=1}^n \left[g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u)), e_a) g(\pi_* X, \xi) \right] \\ & \quad + \sum_{a=1}^n \left[g(\Theta(X, \mathbf{h}(\Theta(\hat{e}_a, \mathbf{h}(\xi))))), e_a \right] - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta(X, \mathbf{h}(\xi))))), e_a \end{aligned}$$

$$+ \sum_{a=1}^n \left[g \left((\nabla_X \Theta)(\hat{e}_a, \mathbf{h}(\xi)), e_a \right) - g \left((\nabla_{\hat{e}_a} \Theta)(X, \mathbf{h}(\xi)), e_a \right) \right] \tag{31}$$

where $\{e_a\}_{a=1, \dots, n}$ is the special g -orthonormal basis sections for π^*TM and Θ is the $(0, 2; 1)$ -tensor on (M, F) given by

$$\begin{aligned} \Theta(X, Y) = & \left(\mathcal{A}(\mathcal{B}(X), \pi_*Y, \bullet) \right)^\# + \left(\mathcal{A}(\mathcal{B}(Y), \pi_*X, \bullet) \right)^\# \\ & - \left(\mathcal{A}(\pi_*X, \pi_*Y, \mathcal{B}(\mathbf{h}(\bullet))) \right)^\# \end{aligned} \tag{32}$$

with the dual section $(\)^\#$ of π^*TM to the Cartan tensor and the $(0, 1; 1)$ -tensor

$$\mathcal{B} := -\frac{1}{F} \frac{\partial}{\partial y^j} \left(\frac{F^2}{2} g^{ir} - y^i y^r \right) \frac{\partial u}{\partial x^r} \frac{\partial}{\partial x^i} \otimes dx^j. \tag{33}$$

Proof. The proof is straightforward from the definitions (0.5) and using the conformal change of the Chern connection given by Theorem 3.

Lemma 6. Let (M, F) be an n -dimensional Finsler manifold. If $\tilde{F} = e^u F$ is Finsler metric conformal to F then the horizontal scalar curvatures $\widetilde{\text{Scal}}_F^H$ and Scal_F^H associated with \tilde{F} and F respectively are conformally related by the equation:

$$\begin{aligned} \widetilde{\text{Scal}}_F^H = & e^{-2u} \left[\text{Scal}_F^H + 2(n-1)\Delta^H u - (n-1)(n-2)\|\nabla u\|_g^2 \right] \\ & + e^{-2u} \left\{ \sum_{b=1}^n (2-n) \left[g(\nabla u, \Theta_{ab}) - g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u)), e_b) \right] \right\} \\ & + e^{-2u} \left\{ \sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta_{ab})), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{ab})), e_a) \right] \right\} \\ & + e^{-2u} \left\{ \sum_{a,b=1}^n \left[g((\nabla_{\hat{e}_b} \Theta)_{ab}, e_a) - g((\nabla_{\hat{e}_a} \Theta)(\hat{e}_b, \hat{e}_b), e_a) \right] \right\} \end{aligned} \tag{34}$$

where $\Theta_{ab} := \Theta(\hat{e}_a, \hat{e}_b)$.

Proof. The proof is straightforward from the definitions (0.5) and using the conformal change of the Chern connection given by Theorem 3.

3.1. Finslerian Ricci Flow

One of the advantages of the Finslerian Ricci tensor obtained by contraction of the Chern hh -curvature is its relation with the second covariant derivative and hence the horizontal Laplacian operators.

Let (M, F_t) be an n -dimensional Finsler manifold of scalar flag curvature and $\{g_t\}_{t \in [0, \tau]}$ the fundamental tensors family of $\{F_t\}_{t \in [0, \tau]}$. We consider the following Finslerian Ricci deformation

$$\frac{\partial}{\partial t} g_t = -2\mathbf{Ric}_{F_t}^H, \quad g_t|_0 = g, \quad \forall t \in [0, \tau), \tag{35}$$

where $\mathbf{Ric}_{F_t}^H$ is the horizontal Ricci tensor defined by (19). The existence of solutions is known in special cases, particularly in Riemannian and Berwald spaces, [3].

3.2. Main Results

We first prove the Theorem 1 on the necessary condition for F_t to be conformal equivalent to F . We find the existence of a family $\{u_t\}_{t \in [0, \tau]}$ of C^∞ functions on M satisfying a parabolic type Equation (2) with initial function equal to zero.

Proof of Theorem 1.1. We denote by g_t and g the fundamental tensors of F_t and F respectively. If $\tilde{F}_t \in [F_t] = [F]$ then there exists a C^∞ function u_t on M such that $\tilde{F}_t = e^{u_t} F$ or equivalently $\tilde{g}_t = e^{2u_t} g$. Then by definition of Finslerian Ricci flow given by relation (1), we have

$$\frac{\partial}{\partial t} \tilde{g}_t = 2e^{2u_t} g \frac{\partial u_t}{\partial t}. \tag{36}$$

By Lemma 31 and using the Equation (36), we obtain

$$\begin{aligned} & \left[e^{2u_t} \left(\frac{\partial u_t}{\partial t} \right) g \right] (\xi, X) \\ &= - \left[\text{Ric}_F^H + \left(\Delta^H u_t - (n-2) \|\nabla u_t\|_g^2 \right) g - (n-2) (H_{u_t} - du_t \circ du_t) \right] (\xi, X) \\ & \quad - (2-n) g(\nabla u_t, \Theta(X, \mathbf{h}(\xi))) + g(\Theta(\mathbf{h}(\nabla u_t), \mathbf{h}(\xi)), \pi_* X) \\ & \quad + g(\Theta(\mathbf{h}(\nabla u_t)), \xi) - \sum_{a=1}^n \left[g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u_t)), e_a) g(\pi_* X, \xi) \right] \\ & \quad - \sum_{a=1}^n \left[g(\Theta(X, \mathbf{h}(\Theta(\hat{e}_a, \mathbf{h}(\xi))))), e_a \right] - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta(X, \mathbf{h}(\xi))))), e_a \\ & \quad - \sum_{a=1}^n \left[g((\nabla_X \Theta)(\hat{e}_a, \mathbf{h}(\xi)), e_a) - g((\nabla_{\hat{e}_a} \Theta)(X, \mathbf{h}(\xi)), e_a) \right] \end{aligned} \tag{37}$$

for every $\xi \in \Gamma(\pi^* TM)$ and for every $X \in \Gamma(T\mathring{TM})$ where $\{e_a\}_{a=1, \dots, n}$ is the special g -orthonormal basis sections for $\pi^* TM$. We have

$$\begin{aligned} & \left[e^{2u_t} \left(\frac{\partial u_t}{\partial t} \right) g \right] (e_b, \pi_* \hat{e}_b) \\ &= - \left[\text{Ric}_F^H + \left(\Delta^H u_t - (n-2) \|\nabla u_t\|_g^2 \right) g - (n-2) (H_{u_t} - du_t \circ du_t) \right] (e_b, \hat{e}_b) \\ & \quad - (2-n) g(\nabla u_t, \Theta_{bb}) + g(\Theta(\mathbf{h}(\nabla u_t), \hat{e}_b), \pi_* \hat{e}_b) + g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) \\ & \quad - \sum_{a=1}^n \left[g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u_t)), e_a) g(\pi_* \hat{e}_b, e_b) \right] - \sum_{a=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\Theta_{ab})), e_a) \\ & \quad + \sum_{a=1}^n g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{bb})), e_a) - \sum_{a=1}^n \left[g((\nabla_{\hat{e}_b} \Theta)_{ab}, e_a) - g((\nabla_{\hat{e}_a} \Theta)_{bb}, e_a) \right] \end{aligned} \tag{38}$$

where $\Theta_{ab} = \Theta(\hat{e}_a, \hat{e}_b)$. We then have

$$\begin{aligned} \frac{\partial u_t}{\partial t} &= -\frac{1}{n} e^{-2u_t} \text{Scal}_F^H + \left(\Delta^H u_t - (n-2) \|\nabla u_t\|_g^2 \right) - \frac{n-2}{n} e^{-2u_t} \left(\Delta^H u_t - \|\nabla u_t\|_g^2 \right) \\ & \quad - \frac{2-n}{n} e^{-2u_t} \sum_{b=1}^n \left[g(\nabla u_t, \Theta_{bb}) - g(\Theta(\mathbf{h}(\nabla u_t), \hat{e}_b), e_b) \right] - \frac{1}{n} K_\Theta \end{aligned} \tag{39}$$

with

$$K_{\Theta} = \sum_{a,b=1}^n \left[g\left(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a\right) - g\left(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{bb})), e_a\right) \right] \\ + \sum_{a,b=1}^n \left[g\left(\left(\nabla_{\hat{e}_b} \Theta\right)(\hat{e}_a, \hat{e}_b), e_a\right) - g\left(\left(\nabla_{\hat{e}_a} \Theta\right)(\hat{e}_b, \hat{e}_b), e_a\right) \right]. \quad (40)$$

From the Equation (39) we obtain the result.

The following Proposition refers to the existence and uniqueness of solution of the Equation (2). We have

Proposition 7. ([8]) Let \mathcal{E} be a bundle of tensors over a smooth compact Riemannian manifold (M, g) . We seek a smooth family $[0, \tau] \rightarrow f_t$ of smooth tensor fields on M ($f_t \in C^\infty(\mathcal{E})$) which satisfies the equation

$$\frac{\partial f}{\partial t} = a^{ij}(t, x, f, \nabla f) \nabla_i \nabla_j f + \psi(t, x, f, \nabla f) \quad \text{with } f(t, x)_{t=0} = \phi(x) \quad (41)$$

$\phi(x)$ is given and belongs to $C^\infty(\mathcal{E})$, the components a^{ij} of a double contravariant symmetric tensor field on M are, in local chart, smooth functions in its arguments, and ψ , with values in \mathcal{E} , is smooth in its components. If the tensor field $a^{ij}(0, x, f, \nabla f)$ is everywhere positive definite, then there exists a unique smooth solution f on $[0, \tau]$ for some $s \leq T$.

From the Proposition 7, we get

Corollary 2. Let (M, F) be an n -dimensional compact Finsler manifold. Assume $n \geq 2$. Then the Equation (2) in Theorem 1 has a unique solution on a parameterized-interval $[0, s]$ for some $s \leq T$.

Proof. We put: $\mathcal{E} = C^\infty(M)$,

$$a^{ij}(t, x, f, \nabla f) \nabla_i \nabla_j f(t) = \frac{2(n-1)}{n} e^{-2u_t} \delta^{ab} \nabla_a \nabla_b u_t$$

and

$$\psi(t, x, f, \nabla f) \\ = -\frac{1}{n} e^{-2u_t} \left[\text{Scal}_F^H - (n-1)(n-2) \|\nabla u_t\|_g^2 \right] \\ - \frac{2-n}{n} e^{-2u_t} \sum_{b=1}^n \left[g(\nabla u_t, \Theta_{ab}) - g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) \right] \\ - \frac{1}{n} e^{-2u_t} \sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{ab})), e_a) \right] \\ - \frac{1}{n} e^{-2u_t} \sum_{a,b=1}^n \left[g\left(\left(\nabla_{\hat{e}_b} \Theta\right)(\hat{e}_a, \hat{e}_b), e_a\right) - g\left(\left(\nabla_{\hat{e}_a} \Theta\right)(\hat{e}_b, \hat{e}_b), e_a\right) \right].$$

Then the relations (41) and (2) are equivalent with $u_t|_{t=0} = 0$.

Moreover,

$$a^{ij}(0, x, f, \nabla f) = \frac{2(n-1)}{n} \delta^{ab},$$

which is positive definite since $\frac{2(n-1)}{n}$ is positive and δ^{ab} is nonnegative.

Applying the Proposition 7 we obtain the result.

When a Finsler metric F is of Einstein type, we obtain the existence on a compact Finsler-Einstein manifold of a Ricci flow which preserves the conformal class.

Proof of Theorem 1.2. For $t \in [0, s)$, $\tilde{g}(t) := e^{2u_t} g$. Since $u_t|_{t=0} = 0$, we have

$$\tilde{g}_t|_{t=0} = e^{2u_0} g = g.$$

It follows that, since u_t is solution of Equation (2),

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{g}(t) &= 2e^{2u_t} g \frac{\partial u_t}{\partial t} = -2 \left\{ \frac{1}{n} \mathbf{Scal}_F^H \pi^* g + \left(\Delta^H u_t - (n-2) \|\nabla u_t\|_g^2 \right) g \right. \\ &\quad + \frac{n-2}{n} \left(\Delta^H u_t + \|\nabla u_t\|_g^2 \right) g + \frac{2-n}{n} \sum_{b=1}^n g(\nabla u_t, \Theta_{ab}) g \\ &\quad + \frac{2}{n} \sum_{b=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) g - \sum_{b=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) g \\ &\quad + \frac{1}{n} \sum_{a,b=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{ab})), e_a) \right] g \\ &\quad \left. + \frac{1}{n} \sum_{a,b=1}^n \left[g((\nabla_{\hat{e}_b} \Theta)(\hat{e}_a, \hat{e}_b), e_a) - g((\nabla_{\hat{e}_a} \Theta)(\hat{e}_b, \hat{e}_b), e_a) \right] g \right\}. \end{aligned} \tag{42}$$

If F is a Finsler-Einstein metric, the trace-free horizontal Ricci tensor associated to F vanishes and then have

$$\mathbf{Ric}_F^H = \frac{1}{n} (\mathbf{Scal}_F^H \pi^* g). \tag{43}$$

If the tensor T_t is conformally equivalent to g , then it holds

$$\begin{aligned} T_{cd}(t) &= \frac{\text{trace } T_t}{n} \delta_{cd} \\ &= \frac{n-2}{n} \left(\Delta^H u_t + \|\nabla u_t\|_g^2 \right) e^{2u_t} \delta_{cd} + \frac{2-n}{n} \sum_{b=1}^n g(\nabla u_t, \Theta_{ab}) e^{2u_t} \delta_{cd} \\ &\quad + \frac{2}{n} \sum_{b=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) e^{2u_t} \delta_{cd} - \sum_{a=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u_t)), e_b) e^{2u_t} \delta_{cd} \\ &\quad + \frac{1}{n} \sum_{a,b=1}^n g(\Theta(\hat{e}_b, \mathbf{h}(\Theta(\hat{e}_a, \hat{e}_b))), e_a) e^{2u_t} \delta_{cd} - \frac{1}{n} \sum_{a,b=1}^n g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{ab})), e_a) e^{2u_t} \delta_{cd} \\ &\quad + \frac{1}{n} \sum_{a,b=1}^n g((\nabla_{\hat{e}_b} \Theta)(\hat{e}_a, \hat{e}_b), e_a) e^{2u_t} \delta_{cd} - \frac{1}{n} \sum_{a,b=1}^n g((\nabla_{\hat{e}_a} \Theta)(\hat{e}_b, \hat{e}_b), e_a) e^{2u_t} \delta_{cd}. \end{aligned} \tag{44}$$

Putting (43) and (44) in (42), we obtain

$$\begin{aligned} \left[\frac{\partial}{\partial t} \tilde{g}(t) \right] (e_b, \pi_* \hat{e}_b) &= -2 \left\{ \mathbf{Ric}_F^H + \left(\Delta^H u_t - (n-2) \|\nabla u_t\|_g^2 \right) \underline{g} + T_t \right\} (e_b, \pi_* \hat{e}_b) \\ &= -2 \left[\mathbf{Ric}_F^H + \left(\Delta^H u - (n-2) \|\nabla u\|_g^2 \right) \underline{g} - (n-2) (H_u - du \circ du) \right] (e_b, \hat{e}_b) \\ &\quad - (2-n) g(\nabla u, \Theta_{bb}) + g(\Theta(\mathbf{h}(\nabla u), \hat{e}_b), \pi_* \hat{e}_b) + g(\Theta(\hat{e}_b, \mathbf{h}(\nabla u)), e_b) \\ &\quad - \sum_{a=1}^n \left[g(\Theta(\hat{e}_a, \mathbf{h}(\nabla u)), e_a) g(\pi_* \hat{e}_b, e_b) \right] \\ &\quad - \sum_{a=1}^n \left[g(\Theta(\hat{e}_b, \mathbf{h}(\Theta_{ab})), e_a) - g(\Theta(\hat{e}_a, \mathbf{h}(\Theta_{bb})), e_a) \right] \end{aligned}$$

$$-\sum_{a=1}^n \left[g \left((\nabla_{\hat{e}_b} \Theta) (\hat{e}_a, \hat{e}_b), e_a \right) - g \left((\nabla_{\hat{e}_a} \Theta) (\hat{e}_b, \hat{e}_b), e_a \right) \right] = -2\widetilde{\mathbf{Ric}}_{\tilde{F}}^H.$$

Hence, $\{\tilde{F}(t)\}_{t \in [0, s]}$ is a Ricci deformation of F , by construction of $[g_t] = [g]$, $\forall t \in [0, s]$. The uniqueness derives from Theorem 1 and Corollary 2.

Remark 3.1. If F is a Riemannian metric then, by the relation (32), the $(0, 2, 1)$ -tensor Θ vanishes. The Theorem 2 becomes the result in [9] for the Riemannian case.

4. Conclusions

In this present work, we use the pulled-back bundle approach [2] to introduce a Finslerian horizontal Ricci flow, called Finslerian Ricci deformation. This approach is natural and is very important because it facilitates the analogy with the Riemannian geometry.

Using the results of the previous sections we plan to study, in the future, the evolution of the Chern hh -curvature and its various traces under the Finslerian Ricci deformation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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