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Three-Qutrit Topological SWAP Logic Gate for ISK (I = 1, S = 1, K = 1) Spin System

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Abstract

Three Zeeman levels of spin-1 electron or nucleus are called as qutrits in quantum computation. Then, ISK (I = 1, S = 1, K = 1) spin system can be represented as three-qutrit states. Quantum circuits and algorithms consist of quantum logic gates. By using SWAP logic gate, two quantum states are exchanged. Topological quantum computing can be applied in quantum error correction. In this study, first, Yang-Baxter equation is modified for ISK (I = 1, S = 1, K = 1) spin system. Then three-qutrit topological SWAP logic gate is obtained. This SWAP logic gate is applied for three-qutrit states of ISK (I = 1, S = 1, K = 1) spin system. Three-qutrit SWAP logic gate is also applied to the product operators of ISK (I = 1, S = 1, K = 1) spin system. For these two applications, expected exchange results are found.

Keywords

Qutrits, SWAP Logic Gate, Yang-Baxter Equation, Product Operators

1. Introduction

A unit of information in quantum information processing is called qubit [1]. Qubits can be represented by two states of any quantum system such as magnetic quantum numbers of spin-1/2 [2]. For spin-1, magnetic quantum levels are called as qutrits [3] [4] [5]. Then, three-qutrit states can be obtained from ISK (I = 1, S = 1, K = 1) spin system. Quantum circuits and algorithms are consisting of quantum logic gates. In order to exchange two quantum states, SWAP logic gate is used for two qubit and two qutrit states [6]-[12]. Some studies on quantum logic gates of qutrits can be found elsewhere (e.g. [13] [14]). In topological quantum computation, anyons are used for storing and manipulating quantum information [15] [16] [17]. Topological quantum computation can be useful in quantum error correction [15]. The Yang-Baxter equation is used to exchange two quantum states in topological quantum computing [16]. As a quantum mechanical method, product operator theory can be useful in NMR quantum computing [6] [7] [18]. For example, SWAP pulse sequence can be applied to the product operators of related spin system (e.g. [19]).

In this study, by using two-qutrit SWAP logic gate, a three-qutrit topological SWAP logic gate is obtained. Then this logic gate is applied for three-qutrit states by using modified Yang-Baxter equation for spin-1. Obtained three-qutrit topological SWAP logic gate is also applied to the product operators of ISK ($I = 1$, $S = 1$, $K = 1$) spin system.

2. Theory

Zeeman levels of spin-1 electron or nucleus are referred as qutrit. For $I = 1$ nucleus, there are three magnetic quantum numbers of 1, 0 and −1. For these magnetic quantum numbers, corresponding qutrit states can be represented as $|0\rangle$, $|1\rangle$ and $|2\rangle$ respectively. In Hilbert space, matrix representations of these qutrit states are given as

$$|0\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad |1\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad |2\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{1}$$

For two spin-1 system such as IS ($I = 1$, $S = 1$) spin system, nine two-qutrit states of $|00\rangle$, $|01\rangle$, $|02\rangle$, $|10\rangle$, $|11\rangle$, $|12\rangle$, $|20\rangle$, $|21\rangle$ and $|22\rangle$ are obtained by direct products of single qutrit states [9]. For example $|01\rangle=|0\rangle\otimes|1\rangle$ is 9 × 1 matrix. Two-qutrit CNOT gates can be found by using the ternary addition of qutrit states:

$$\mathrm{CNOT_a}(T)|a,b\rangle=|a,b\oplus a\rangle, \tag{2a}$$

$$\mathrm{CNOT_b}(T)|a,b\rangle=|a\oplus b,b\rangle. \tag{2b}$$

Here T is used for ternary. These two-qutrit CNOT gates are 9 × 9 matrices and they can be written in Dirac notation as following:

$$\begin{aligned}\mathrm{CNOT_a}(T)=&|00\rangle\langle 00|+|01\rangle\langle 01|+|02\rangle\langle 02|+|10\rangle\langle 11|+|11\rangle\langle 12|\\&+|12\rangle\langle 10|+|20\rangle\langle 22|+|21\rangle\langle 20|+|22\rangle\langle 21|\end{aligned}, \tag{3a}$$

$$\begin{aligned}\mathrm{CNOT_b}(T)=&|00\rangle\langle 00|+|01\rangle\langle 11|+|02\rangle\langle 22|+|10\rangle\langle 10|+|11\rangle\langle 21|\\&+|12\rangle\langle 02|+|20\rangle\langle 20|+|21\rangle\langle 01|+|22\rangle\langle 12|\end{aligned}. \tag{3b}$$

By using the SWAP logic gate two quantum states are exchanged as following:

$$\mathrm{SWAP}|ab\rangle=|ba\rangle. \tag{4}$$

For two-qubit states of $|ab\rangle$, SWAP quantum logic gate can be obtained by using two qubit $\mathrm{CNOT_a}$ and $\mathrm{CNOT_b}$ logic gates as following [1] [2]:

$$(\mathrm{CNOT_a})(\mathrm{CNOT_b})(\mathrm{CNOT_a})=(\mathrm{CNOT_b})(\mathrm{CNOT_a})(\mathrm{CNOT_b}). \tag{5}$$

This is not valid for two qudit states. Different implementations of SWAP logic gate for two qudit states are suggested in the literature (e.g. [8] [20]). By using one of these implementations, two qudit SWAP logic gate can be written as [8]

$$\left[I\otimes(-I)\right]\mathrm{CNOT_a}\left[(-I)\otimes I\right]\mathrm{CNOT_b}\left[(-I)\otimes I\right]\mathrm{CNOT_a}. \tag{6}$$

where, I is 3 × 3 unity matrix for two-qutrit states. By using this equation together with the Equations (3a) and (3b), two-qutrit SWAP logic gate in Dirac notation can be easily obtained:

$$\begin{aligned}\mathrm{SWAP}(T)=&|00\rangle\langle 00|+|01\rangle\langle 10|+|02\rangle\langle 20|+|10\rangle\langle 01|+|11\rangle\langle 11|\\&+|12\rangle\langle 21|+|20\rangle\langle 02|+|21\rangle\langle 12|+|22\rangle\langle 22|\end{aligned}. \tag{7}$$

Diagrammatical representation of Yang-Baxter equation for a three-qutrit state is shown in **Figure 1**. Yang-Baxter equation is [13]

$$(R\otimes I)(I\otimes R)(R\otimes I)=(I\otimes R)(R\otimes I)(I\otimes R). \tag{8}$$

In this equation, we can use R = SWAP(T). In this case input is $|abc\rangle$ and then output is $|cba\rangle$. When we use three-qutrit states, this figure can be used as three-qutrit topological SWAP logic gate. Also this logic gate can be used for the reversal of the qubits or qudits.

3. Results and Discussion

For ISK (I = 1, S = 1, K = 1) spin system, 27 three-qutrit states of $|00\rangle 0$, $|001\rangle$, $|002\rangle$, $|010\rangle$, $|011\rangle$, $|012\rangle$, $|020\rangle$, $|021\rangle$, $\cdots$, $|222\rangle$ are obtained by direct products of single qutrit states. In Yang-Baxter Equation for three-qutrit (Equation (8)), R is two-qutrit SWAP logic gate as given in Equation (7). Then, the result of Yang-Baxter Equation for three-qutrit, U is obtained as 27 × 27 matrix. So, this can be called as three-qutrit topological SWAP logic gate. The matrix representation of this three-qutrit SWAP logic gate in Dirac notation is

$$\begin{aligned}U=&|000\rangle\langle 000|+|001\rangle\langle 100|+|002\rangle\langle 200|+|010\rangle\langle 010|+|011\rangle\langle 110|+|012\rangle\langle 210|\\&+|020\rangle\langle 020|+|021\rangle\langle 120|+|022\rangle\langle 220|+|100\rangle\langle 001|+|101\rangle\langle 101|\\&+|102\rangle\langle 201|+|110\rangle\langle 011|+|111\rangle\langle 111|+|112\rangle\langle 211|+|120\rangle\langle 021|+|121\rangle\langle 121|\\&+|122\rangle\langle 221|+|200\rangle\langle 002|+|201\rangle\langle 102|+|202\rangle\langle 202|+|210\rangle\langle 012|\\&+|211\rangle\langle 112|+|212\rangle\langle 212|+|220\rangle\langle 022|+|221\rangle\langle 122|+|222\rangle\langle 222|\end{aligned} \tag{9}$$

When the U matrix is applied to three-qutrit states, three-qutrit topological SWAP is performed as given in **Table 1**.

Nine Cartesian spin angular momentum operators for I = 1 are E_I, I_x, I_y, I_z, I_z^2, $[I_x,I_z]_+$, $[I_y,I_z]_+$, $[I_x,I_y]_+$ and $(I_x^2-I_y^2)$ [21]. Similarly, there are

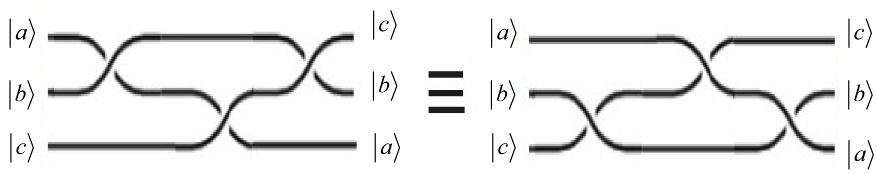

Figure 1. Diagrammatical representations of Yang-Baxter equation for a three-qutrit state. In both sides, there are three two-qutrit SWAP logic gate in different order.

also nine Cartesian spin angular momentum operators for both $S = 1$ and $K = 1$ spins. So, $9\times 9\times 9 = 729$ product operators are obtained with direct products of these spin angular momentum operators for ISK ($I = 1$, $S = 1$, $K = 1$) spin system. These product operators for ISK ($I = 1$, $S = 1$, $K = 1$) spin system are 27×27 matrices. A Hamiltonian, H can be applied to a product operator as following:

$$UPU^{\dagger} = Q . \tag{10}$$

where, $U = \exp(-iHt)$. In this study U will be three-qutrit SWAP logic gate as given in Equation (9). The SWAP operation can be applied to any product operator for ISK ($I = 1$, $S = 1$, $K = 1$) spin system. For example, when the SWAP operation applied to $I_y S_z^2 K_z$ product operator, $I_z S_z^2 K_y$ is obtained:

$$UI_y S_z^2 K_z U^{\dagger} = I_z S_z^2 K_y . \tag{11}$$

Similar effects of the SWAP operation for the remaining product operators can be found. The effects of the SWAP operation for some product operators are presented in Table 2. As shown in Table 2, the expected SWAP operation is performed for the product operators of ISK ($I = 1$, $S = 1$, $K = 1$) spin system.

Table 1. Application of three-qutrit topological SWAP logic gate.

Input, $\|abc\rangle$	Output, $U\|abc\rangle = \|cba\rangle$
$\|000\rangle$	$\|000\rangle$
$\|001\rangle$	$\|100\rangle$
$\|002\rangle$	$\|200\rangle$
$\|010\rangle$	$\|010\rangle$
$\|011\rangle$	$\|110\rangle$
$\|012\rangle$	$\|210\rangle$
$\vdots$	$\vdots$
$\|220\rangle$	$\|022\rangle$
$\|221\rangle$	$\|122\rangle$
$\|222\rangle$	$\|222\rangle$

Table 2. Some product operators P and Q for ISK ($I = 1$, $S = 1$, $K = 1$) spin system before and after the SWAP operation, respectively.

Product operator, P	Product operator, Q
$I_y \otimes E_S \otimes E_K$	$E_I \otimes E_S \otimes K_y$
$E_I \otimes S_y^2 \otimes E_K$	$E_I \otimes S_y^2 \otimes E_K$
$I_y \otimes S_z^2 \otimes E_K$	$E_I \otimes S_z^2 \otimes K_y$
$E_I \otimes S_z^2 \otimes K_y^2$	$I_y^2 \otimes S_z^2 \otimes E_K$
$I_x \left(S_x^2 - S_y^2\right)\left(K_x^2 - K_y^2\right)$	$\left(I_x^2 - I_y^2\right)\left(S_x^2 - S_y^2\right)K_x$
$I_y^2 [S_x, S_z]_+ K_z^2$	$I_z^2 [S_x, S_z]_+ K_y^2$
$[I_x, I_z]_+ S_y [K_x, K_y]_+$	$[I_x, I_y]_+ S_y [K_x, K_z]_+$
$\left(I_x^2 - I_y^2\right)[S_x, S_z]_+ [K_x, K_z]_+$	$[I_x, I_z]_+ [S_x, S_z]_+ \left(K_x^2 - K_y^2\right)$

4. Conclusion

Three magnetic quantum numbers of spin-1 are called as qutrits. Then ISK ($I = 1$, $S = 1$, $K = 1$) spin system can be used as three-qutrit states. In this study, first, Yang-Baxter equation is modified for qutrits. Then, three-qutrit topological SWAP logic gate is suggested and applied by using this modified equation. Three-qutrit SWAP logic gate is also applied to the product operators for ISK ($I = 1$, $S = 1$, $K = 1$) spin system. Expected exchange results are obtained for three-qutrit states and for the product operators of ISK ($I = 1$, $S = 1$, $K = 1$) spin system.

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