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Mathematical Analysis and Simulation of an Age-Structured Model of Two-Patch for Tuberculosis (TB)

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Abstract

This paper studied a structured model by age of tuberculosis. A population divided into two parts was considered for the study. Each subpopulation is submitted to a program of vaccination. It was allowed the migration of vaccinated people only between the two patches. After the determination of $\Re(\psi)$ and $\Re_0$, the local and global stability of the disease-free equilibrium was studied. It showed the existence of three endemic equilibrium points. The theoretical results were illustrated by a numeric simulation.

Keywords

Age-Structured, Reproductive Number, Two-Patch, TB, Stability, Simulation

1. Introduction

Tuberculosis (TB) (short for tubercle bacillus) is a widespread, infectious disease caused by various strains of mycobacteria, usually *Mycobacterium tuberculosis* (MTB). Tuberculosis typically attacks the lungs, but can also affect other parts of the body [1]. To be infected bacilli must penetrate deep into the alveoli, but the contagiousness of the disease is relatively low and depends on the immune system of subjects. Individuals at highest risk are young children, adults, deficient elderly, and people living in precarious socio-economic conditions, in nursing or whose immunity is deficient (AIDS, immunosuppressive therapy ...) [2]. This is one of the most common old infectious diseases [3] [4], with about two billion people being currently infected. There are

about nine million new cases of infection each year and two million deaths per year according to WHO estimations [3] [5]. For more information, many authors have worked on the epidemiology of tuberculosis [1]-[3] [5]-[13]. In many developing countries in general and sub-Saharan Africa particularly, TB is the leading cause of death, accounting for about two million deaths and a quarter of avoidable adult deaths [11].

It is well known that factors such as the emergence of drug resistance against tuberculosis, the growth of the incidence of HIV in recent years, as well as other diseases favor the development of Koch bacillus in the body call for improved strategies to control this deadly disease [2] [10] [14]. Last May, the World Health Assembly approved an ambitious strategy for 20 years (2016-2035) to put an end to World TB epidemic (World Day of fight against tuberculosis—March 24, 2015). In literature, several articles discussed about coinfection: TB-HIV/AIDS and the most recent is [2]. Nowadays, it is not a secret for everyone that fighting against infectious diseases is also a fight against poverty. Humans are traditionally organized into well-defined social units, such as families, tribes, villages, cities, countries or regions are good examples of patches [11] [12]. For this study, two subpopulations were considered and each was subjected to a vaccination program. However, only the vaccinated individuals can migrate from one patch to another. Despite that we have neglected the relapse rate, to avoid any risk of treated individuals' reactivation, any migration between patches was allowed. After proving that the problem is well defined and it has a unique solution if the initial condition is given, we are able to calculate the reproduction of numbers $\Re(\psi)$ and $\Re_0$. We have established the existence conditions for three endemic equilibrium points, and the conditions of local and global stability of the equilibrium point without disease. Finally, numerical simulations illustrate clinical outcomes. This paper is organized as follows: Section 2 introduces the two-patch model structured in age to study the dynamics of TB transmission. The existence of positive and unique solutions is demonstrated in Section 3. The point of equilibrium without disease, reproductive numbers $\Re(\psi)$ and $\Re_0$ are defined in the section 4 with the local and global stability of the disease-free equilibrium point. The existence of three endemic equilibrium points is proven in Section 5. Some numerical simulation results are given in Section 6. In Section 7, we have a discussion, conclusion and further work.

2. Parameters and Mathematical Model Formulation

Two-patch age structured model of tuberculosis was considered. The model is to split the population into two subpopulations. The recruitment is only possible in the class of susceptible and the vaccinated individuals were able to migrate between the two subpopulations. Each subpopulation is divided into five classes based on their epidemiological status: susceptible, vaccinated, latent, infectious or treated. We denote these subgroups $S_i(t,a)$, $v_i(t,a)$, $L_i(t,a)$, $I_i(t,a)$ and $J_i(t,a)$ respectively. The birth rate of the patch i is $b_i(a)$; $\mu_i(a)$ and $\mu(a)$ denote the mortality rate related to the

disease relative to the patch i and the rate of natural mortality. The time and age depended of the force of infection of the subpopulation i is $\lambda_i(t,a)$ and vaccination rate is $\psi_i(a)$; $p_i(a,a')$ is the probability that an infective individual of age a' will have contact with and successfully infect a susceptible individual of age a, $c_i(a)$ is the age-specic per-capita contact/activity rate (all of these functions are assumed to be continuous and to be zero beyond some maximum age). A fraction ϕ_i of newly infected individuals of the sub-population i is assumed to undergo a fast progression directly to the infectious class I_i. Rates of migration, of susceptible passage to latent infectious state and treatment are respectively ρ_i; k_i and r_i. Risk reduction rates of treatment and vaccination are σ_i and δ_i respectively, $0 \le \sigma_i \le (1-\phi_i)$, $0 \le \delta_i \le (1-\phi_i)$, in this paper $i = 1,2$.

The age-structured model for the transmission of TB (see Figure 1) is described by the following system of partial differential equations:

$$\begin{cases}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_1(t,a) = b_1(a)N(t,a) - \left[\lambda_1(t,a)+\psi_1(a)+\mu(a)\right]S_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)L_1(t,a) = \lambda_1(t,a)\left[(1-\phi_1)S_1(t,a)+\sigma_1 J_1(t,a)+\delta_1 V_1(t,a)\right] - \left(k_1+\mu(a)\right)L_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_1(t,a) = k_1 L_1(t,a) - \left(r_1+\mu(a)+\mu_1(a)\right)I_1(t,a) + \phi_1\lambda_1(t,a)S_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)J_1(t,a) = r_1 I_1(t,a) - \left(\sigma_1\lambda_1(t,a)+\mu(a)\right)J_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)V_1(t,a) = \psi_1(a)S_1(t,a) + \rho_2 V_2(t,a) - \left(\rho_1+\mu(a)+\delta_1\lambda_1(t,a)\right)V_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)S_2(t,a) = b_2(a)N(t,a) - \left[\lambda_2(t,a)+\psi_2(a)+\mu(a)\right]S_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)L_2(t,a) = \lambda_2(t,a)\left[(1-\phi_2)S_2(t,a)+\sigma_2 J_2(t,a)+\delta_2 V_2(t,a)\right] - \left(k_2+\mu(a)\right)L_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)I_2(t,a) = k_2 L_2(t,a) - \left(r_2+\mu(a)+\mu_2(a)\right)I_2(t,a) + \phi_2\lambda_2(t,a)S_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)J_2(t,a) = r_2 I_2(t,a) - \left(\sigma_2\lambda_2(t,a)+\mu(a)\right)J_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)V_2(t,a) = \psi_2(a)S_2(t,a) + \rho_1 V_1(t,a) - \left(\rho_2+\mu(a)+\delta_2\lambda_2(t,a)\right)V_2(t,a)
\end{cases} \tag{1}$$

with initial and boundary conditions:

$$\begin{cases}
S_i(t,0) = \int_{a_1}^{a_2} b_i(a)N(t,a)\,\mathrm{d}a \\
L_i(t,0) = V_i(t,0) = I_i(t,0) = J_i(t,0) = 0 \\
S_i(0,a) = S_{0i}(a); L_i(0,a) = L_{0i}(a); V_i(0,a) = V_{0i}(a) \\
I_i(0,a) = I_{0i}(a); J_i(0,a) = J_{0i}(a)
\end{cases}$$

and $\lambda_i(t,a) = \beta_i(a)c_i(a)\int_0^{a_+} \frac{I_i(t,a')}{N(t,a')} p_i(a,a')\,\mathrm{d}a'$, assume that assume that

$$p_i(a,a') = g_i(a)\hat{\beta}_i(a') \tag{2}$$

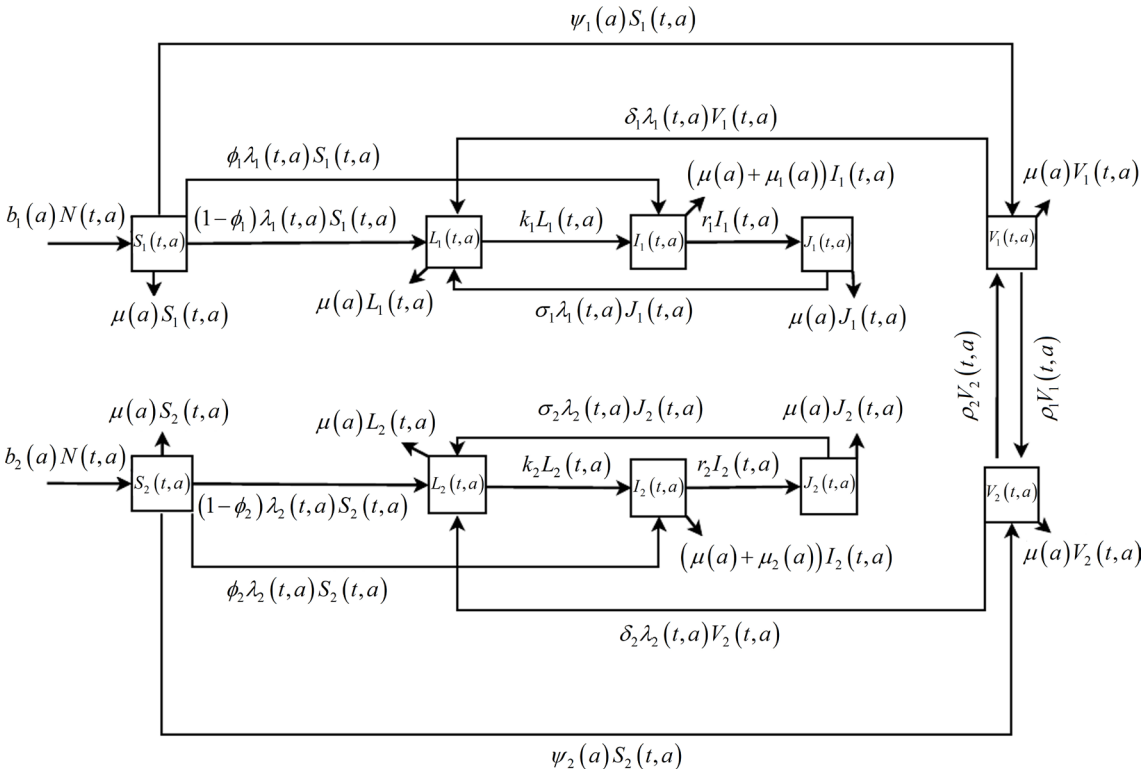

Figure 1. Flow chart of the two-patch model for tuberculosis disease transmission.

(see Greenhalgh, 1988 [15] and Dietz Schenzle, 1985 [16]), and

$$N(t,a)=S_1(t,a)+L_1(t,a)+I_1(t,a)+J_1(t,a)+V_1(t,a)+S_2(t,a)+L_2(t,a)+I_2(t,a)+J_2(t,a)+V_2(t,a).$$

By summing equations of system (1) and (2), we obtain the following equations for the total population $N(t,a)$:

$$\begin{cases}\left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)N(t,a)=\big(b(a)-\mu(a)\big)N(t,a)-\mu_1(a)I_1(t,a)-\mu_2(a)I_2(t,a)\\ N(t,0)=\int_{a_1}^{a_2}b(a)N(t,a)\,\mathrm{d}a\end{cases}\tag{3}$$

where $b(a)=b_1(a)+b_2(a)$; a_1 and a_2 are respectively the minimum and maximum age of procreation and a_+ is the maximum age of an individual, with $a_+<+\infty$.

Let

$$\begin{cases}s_i(t,a)=\dfrac{S_i(t,a)}{N(t,a)};l_i(t,a)=\dfrac{L_i(t,a)}{N(t,a)}\\ i_i(t,a)=\dfrac{I_i(t,a)}{N(t,a)}\\ j_i(t,a)=\dfrac{J_i(t,a)}{N(t,a)};v_i(t,a)=\dfrac{V_i(t,a)}{N(t,a)}.\end{cases}\tag{4}$$

The system (1) can be normalized as the following system:

$$
\begin{cases}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_1(t,a)=b_1(a)-\left[\lambda_1(t,a)+\psi_1(a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right]s_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)l_1(t,a)=\lambda_1(t,a)\left[(1-\phi_1)s_1(t,a)+\sigma_1 j_1(t,a)+\delta_1 v_1(t,a)\right]-\left(k_1+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)l_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_1(t,a)=-\left(r_1+\mu_1(a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)i_1(t,a)+\phi_1\lambda_1(t,a)s_1(t,a)+k_1l_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)j_1(t,a)=r_1i_1(t,a)-\left(\sigma_1\lambda_1(t,a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)j_1(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)v_1(t,a)=-\left(b(a)+\rho_1+\delta_1\lambda_1(t,a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)v_1(t,a)+\psi_1(a)s_1(t,a)+\rho_2v_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)s_2(t,a)=b_2(a)-\left[\lambda_2(t,a)+\psi_2(a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right]s_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)l_2(t,a)=\lambda_2(t,a)\left[(1-\phi_2)s_2(t,a)+\sigma_2 j_2(t,a)+\delta_2 v_2(t,a)\right]-\left(k_2+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)l_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)i_2(t,a)=-\left(r_2+\mu_2(a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)i_2(t,a)+\phi_2\lambda_2(t,a)s_2(t,a)+k_2l_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)j_2(t,a)=r_2i_2(t,a)-\left(\sigma_2\lambda_2(t,a)+b(a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)j_2(t,a)\\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)v_2(t,a)=-\left(b(a)+\rho_2+\delta_2\lambda_2(t,a)-\mu_1(a)i_1(t,a)-\mu_2(a)i_2(t,a)\right)v_2(t,a)+\psi_2(a)s_2(t,a)+\rho_1v_1(t,a)
\end{cases}
\tag{5}
$$

with boundary conditions

$$s_i(t,0)=\Lambda_i;v_i(t,0)=l_i(t,0)=i_i(t,0)=j_i(t,0)=0$$

with $\Lambda_1+\Lambda_2=1$. The problem is well-posedness, the methode of proof is the same used in [8].

3. Existence of Positive Solutions

In this section we will prove that the system (5) has a unique positive solution, and to achieve this we will write the system (5) in compact form (abstract Cauchy problem).

Consider the Banach space X defined by $X=\left(L^1(0,a_+)\right)^{10}$, endowed with the norm

$$\|\varphi\|=\sum_{i=1}^{2}\sum_{j=1}^{5}\|\varphi_{ij}\| \tag{6}$$

where

$\varphi(a)=(\varphi_{11}(a),\varphi_{12}(a),\varphi_{13}(a),\varphi_{14}(a),\varphi_{15}(a),\varphi_{21}(a),\varphi_{22}(a),\varphi_{23}(a),\varphi_{24}(a),\varphi_{25}(a))^{\mathrm{T}}\in X$

and $\|.\|$ is the norm of $L^1(0,a_+)$. Let

$$\Omega=\left\{(s_1,l_1,i_1,j_1,v_1,s_2,l_2,i_2,j_2,v_2)\in X_+\setminus 0\le s_1+l_1+i_1+j_1,v_1+s_2+l_2+i_2+j_2+v_2\le 1\right\} \tag{7}$$

The state space of system (5), where $X_+=\left(L^1_+(0,a_+)\right)^{10}$, and $L^1_+(0,a_+)$ denotes the positive cone of $L^1(0,a_+)$. Let A be a linear operator defined by

$$(A\varphi)(a)=(A_{11},A_{12},A_{13},A_{14},A_{15},A_{21},A_{22},A_{23},A_{24},A_{25})^{\mathrm{T}}. \tag{8}$$

To determine the components A_{ij}, we neglect terms of order two and those which

are not multiplied by s_i, l_i, i_i, j_i or v_i in system (5) (see [17]), we obtain:

$$(a)\begin{cases}\dfrac{\partial}{\partial t}i_1(t,a)=-\dfrac{\partial}{\partial a}i_1(t,a)-\big(r_1+\mu_1(a)+b(a)\big)i_1(t,a)+k_1l_1(t,a)\\ \dfrac{\partial}{\partial t}j_1(t,a)=-\dfrac{\partial}{\partial a}j_1(t,a)+r_1i_1(t,a)-b(a)\,j_1(t,a)\\ \dfrac{\partial}{\partial t}v_1(t,a)=-\dfrac{\partial}{\partial a}v_1(t,a)-\big(b(a)+\rho_1\big)v_1(t,a)+\psi_1(a)s_1(t,a)+\rho_2v_2(t,a)\\ \dfrac{\partial}{\partial t}s_2(t,a)=-\dfrac{\partial}{\partial a}s_2(t,a)-\big[\psi_2(a)+b(a)\big]s_2(t,a)\\ \dfrac{\partial}{\partial t}l_2(t,a)=-\dfrac{\partial}{\partial a}l_2(t,a)-\big(k_2+b(a)\big)l_2(t,a)\\ \dfrac{\partial}{\partial t}i_2(t,a)=-\dfrac{\partial}{\partial a}i_2(t,a)-\big(r_2+\mu_i(a)+b(a)\big)i_2(t,a)+k_2l_2(t,a)\\ \dfrac{\partial}{\partial t}j_2(t,a)=-\dfrac{\partial}{\partial a}j_2(t,a)+r_2i_2(t,a)-b(a)\,j_2(t,a)\\ \dfrac{\partial}{\partial t}v_2(t,a)=-\dfrac{\partial}{\partial a}v_2(t,a)-\big(b(a)+\rho_2\big)v_2(t,a)+\psi_2(a)s_2(t,a)+\rho_1v_1(t,a).\end{cases}$$

After replacing s_1, l_1, i_1, j_1, v_1, s_2, l_2, i_2, j_2 and v_2 by $\varphi_{11}(a)$, $\varphi_{12}(a)$, $\varphi_{13}(a)$, $\varphi_{14}(a)$, $\varphi_{15}(a),\varphi_{21}(a)$, $\varphi_{22}(a)$, $\varphi_{23}(a)$, $\varphi_{24}(a)$, $\varphi_{25}(a)$ in the system (a) respectively, the coordinates of A_{ij} are obtained from straight expressions (note that each

$A_{ij}=\big(f(\varphi_{11}),f(\varphi_{12}),f(\varphi_{13}),f(\varphi_{14}),f(\varphi_{15}),f(\varphi_{21}),f(\varphi_{22}),f(\varphi_{23}),f(\varphi_{24}),f(\varphi_{25})\big)$) with respect to φ_{ij} are given by:

$$\begin{cases}A_{12}=\left(0,-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{12}-\big(b(a)+k_1\big)\varphi_{12},0,0,0,0,0,0,0,0\right)\\ A_{13}=\left(0,k_1\varphi_{12},-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{13}-\big(r_1+\mu_1(a)+b(a)\big)\varphi_{13},0,0,0,0,0,0,0\right)\\ A_{14}=\left(0,0,r_1\varphi_{13},-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{14}-b(a)\varphi_{14},0,0,0,0,0,0\right)\\ A_{15}=\left(\psi_1(a)\varphi_{11},0,0,0,-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{15}-\big(\rho_1+b(a)\big)\varphi_{15},0,0,0,0,\rho_2\varphi_{25}\right)\\ A_{21}=\left(0,0,0,0,0,-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{21}-\big(\psi_2(a)+b(a)\big)\varphi_{21},0,0,0,0\right)\\ A_{22}=\left(0,0,0,0,0,0,-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{22}-\big(b(a)+k_2\big)\varphi_{22},0,0,0\right)\\ A_{23}=\left(0,0,0,0,0,0,k_2\varphi_{22},-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{23}-\big(r_2+\mu_2(a)+b(a)\big)\varphi_{23},0,0\right)\\ A_{24}=\left(0,0,0,0,0,0,0,r_2\varphi_{23},-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{24}-b(a)\varphi_{24},0\right)\\ A_{25}=\left(0,0,0,0,\rho_1\varphi_{15},\psi_2(a)\varphi_{21},0,0,0,-\dfrac{\mathrm{d}}{\mathrm{d}a}\varphi_{25}-\big(\rho_2+b(a)\big)\varphi_{25}\right)\end{cases}. \tag{9}$$

With

$$\varphi(a)=\big(\varphi_{11}(a),\varphi_{12}(a),\varphi_{13}(a),\varphi_{14}(a),\varphi_{15}(a),\varphi_{21}(a),\varphi_{22}(a),\varphi_{23}(a),\varphi_{24}(a),\varphi_{25}(a)\big)^{\mathrm{T}}\in D(A)$$

where $D(A)$ is the domain given by:

$$D(A)=\left\{\varphi\in X \setminus \varphi_{ij}\in AC[0,a_+),\varphi(0)=(\Lambda_1,0,0,0,0,0,\Lambda_2,0,0,0,0)^{\mathrm{T}}\right\}.$$

And $AC[0,a_+)$ denotes the set of absolutely continuous functions on $[0,a_+)$. We also define a nonlinear operator $F:X\to X$ by:

$$(F\varphi)(a)=\begin{pmatrix} b_1(a)-((Q_1\varphi_{13})(a))\varphi_{11}+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{11}\\ ((Q_1\varphi_{13})(a))((1-\varphi_1)\varphi_{11}+\sigma_1\varphi_{14}+\delta_1\varphi_{15})+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{12}\\ \varphi_1((Q_1\varphi_{13})(a))\varphi_{11}+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{13}\\ (\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})-\delta_1((Q_1\varphi_{13})(a))\varphi_{14}\\ (\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})-\sigma_1((Q_1\varphi_{13})(a))\varphi_{15}\\ b_2(a)-((Q_2\varphi_{23})(a))\varphi_{21}+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{21}\\ ((Q_2\varphi_{23})(a))((1-\varphi_2)\varphi_{21}+\sigma_2\varphi_{24}+\delta_2\varphi_{25})+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{22}\\ \varphi_2((Q_2\varphi_{23})(a))\varphi_{21}+(\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})\varphi_{23}\\ (\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})-\delta_2((Q_2\varphi_{23})(a))\varphi_{24}\\ (\mu_1(a)\varphi_{13}+\mu_2(a)\varphi_{23})-\sigma_2((Q_2\varphi_{23})(a))\varphi_{25} \end{pmatrix} \tag{10}$$

where Q_i is a bounded linear operator on $L^1(0,a_+)$ given by

$$(Q_i f)(a)=c_i(a)\beta_i(a)g_i(a)\int_0^{a_+}\hat{\beta}_i(a')f(a')\mathrm{d}a'. \tag{11}$$

Let

$$u(t)=(s_1(.,t),l_1(.,t),i_1(.,t),j_1(.,t),v_1(.,t),s_2(.,t),l_2(.,t),i_2(.,t),j_2(.,t),v_2(.,t))$$

thus, we can rewrite the system (5) as an abstract Cauchy problem:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}t}u(t)=Au(t)+F(u(t))\\ u(0)=u_0\end{cases} \tag{12}$$

where

$$u_0(a)=(s_{01}(a),l_{01}(a),i_{01}(a),j_{01}(a),v_{01}(a),s_{02}(a),l_{02}(a),i_{02}(a),j_{02}(a),v_{02}(a))^{\mathrm{T}}.$$

According to these results we have the following results (see [17]-[19]):

Lemma 1. *The operator F is continuously Fréchet differentiable on X.*

Lemma 2. *The operator A generates a C_0-semigroup of the bounded linear operators e^{tA} and the space Ω is positively invariant by e^{tA}.*

Theorem 1. *For each $u_0\in X_+$ there are a maximal interval of existence $[0,t_{\max})$ and a unique continuous mild solution $u(t,u_0)\in X_+$, $t\in[0,t_{\max})$ for (12) such that*

$$u(t)=u_0\mathrm{e}^{tA}+\int_0^t\mathrm{e}^{A(t-\xi)}F(u(\xi))\mathrm{d}\xi$$

Proof. The proof of this theorem can be found in [18]-[20]. □

4. The Disease-Free Steady State

4.1. Determination of the Disease-Free Equilibrium

A steady state $(s_1(a),l_1(a),i_1(a),j_1(a),v_1(a),s_2(a),l_2(a),i_2(a),j_2(a),v_2(a))$ of sys-

tem (5) must satisfy the following time-independent system of ordinary differential equations:

$$\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}a}s_1(a) = b_1(a) - \left[\beta_1(a)c_1(a)g_1(a)\Gamma_1 + \psi_1(a) + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right]s_1(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}l_1(a) = \beta_1(a)c_1(a)g_1(a)\Gamma_1\left[(1-\phi_1)s_1(a) + \delta_1 v_1(a) + \sigma_1 j_1(a)\right] \\
\qquad\qquad - \left(b(a) + k_1 - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right)l_1(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}i_1(a) = k_1 l_1(a) + \phi_1\beta_1(a)c_1(a)g_1(a)\Gamma_1 s_1(a) - \left[r_1 + b(a) + \mu_1(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right]i_1(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}j_1(a) = r_1 i_1(a) - \left(\sigma_1\beta_1(a)c_1(a)g_1(a)\Gamma_1 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right)j_1(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}v_1(a) = \psi_1(a)s_1(a) + \rho_2 v_2(a) - \left(\delta_1\beta_1(a)c_1(a)g_1(a)\Gamma_1 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_1\right)v_1(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}s_2(a) = b_2(a) - \left[\beta_2(a)c_2(a)g_2(a)\Gamma_2 + \psi_2(a) + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right]s_2(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}l_2(a) = \beta_2(a)c_2(a)g_2(a)\Gamma_2\left[(1-\phi_2)s_2(a) + \delta_2 v_2(a) + \sigma_2 j_2(a)\right] \\
\qquad\qquad - \left(b(a) + k_2 - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right)l_2(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}i_2(a) = k_2 l_2(a) + \phi_2\beta_2(a)c_2(a)g_2(a)\Gamma_2 s_2(a) - \left[r_2 + b(a) + \mu_2(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right]i_2(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}j_2(a) = r_2 i_2(a) - \left(\sigma_2\beta_2(a)c_2(a)g_2(a)\Gamma_2 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a)\right)j_2(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}v_2(a) = \psi_2(a)s_2(a) + \rho_1 v_1(a) - \left(\delta_2\beta_2(a)c_2(a)g_2(a)\Gamma_2 + b(a) - \mu_1(a)i_1(a) - \mu_2(a)i_2(a) + \rho_2\right)v_2(a) \\
\Gamma_i = \int_0^{a_+}\hat{\beta}_i(a)i_i(a)\mathrm{d}a
\end{cases} \tag{13}$$

with initial value conditions

$$s_i(0) = \Lambda_i;\ l_i(0) = i_i(0) = j_i(0) = v_i(0) = 0.$$

Therefore, we obtain the disease-free steady state

$$\begin{cases}
s_i^0(a) = \Lambda_i \mathrm{e}^{-\int_0^a (b(\tau)+\psi_i(\tau))\mathrm{d}\tau} + \int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+\psi_i(\tau))\mathrm{d}\tau} b_i(\eta)\mathrm{d}\eta \\
v_i^0(a) = \Lambda_i - s_i^0(a);\ l_i^0(a) = i_i^0(a) = j_i^0(a) = 0
\end{cases}. \tag{14}$$

4.2. Calculation of the Reproduction Numbers $\Re(\psi)$ - $\Re_0$

To study the stability of the disease-free steady state, we denote the perturbations of system by

$$\begin{cases}
s_i(t,a) = \bar{s}_i(t,a) + s_i^0(a) \\
l_i(t,a) = \bar{l}_i(t,a) + l_i^0(a) \\
i_i(t,a) = \bar{i}_i(t,a) + i_i^0(a) \\
j_i(t,a) = \bar{j}_i(t,a) + j_i^0(a) \\
v_i(t,a) = \bar{v}_i(t,a) + v_i^0(a)
\end{cases}. \tag{15}$$

The perturbations satisfy the following equations:

$$\begin{cases}
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{s}_1(t,a) = -\left[\bar{\gamma}_1(t)\beta_1(a)c_1(a)g_1(a)-\mu_1(a)\bar{i}_1(t,a)-\mu_2(a)\bar{i}_2(t,a)\right]s_1^0(a) \\
\qquad -\left(b(a)+\psi_1(a)\right)\bar{s}_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{l}_1(t,a) = -\left(b(a)+k_1\right)\bar{l}_1(t,a) \\
\qquad +\bar{\gamma}_1(t)\beta_1(a)c_1(a)g_1(a)\left[(1-\phi_1)s_1^0(a)+\delta_1 v_1^0(a)\right] \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{i}_1(t,a) = k_1\bar{l}_1(t,a)+\phi_1\beta_1(a)c_1(a)g_1(a)\bar{\gamma}_1(t)s_1^0(a) \\
\qquad -\left(r_1+\mu_1(a)+b(a)\right)\bar{i}_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{j}_1(t,a) = r_1\bar{i}_1(t,a)-b(a)\bar{j}_1(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{v}_1(t,a) = \psi_1(a)\bar{s}_1(t,a)+\rho_2\bar{v}_2(t,a)-\left(\rho_1+b(a)\right)\bar{v}_1(t,a) \\
\qquad -\left(\delta_1\beta_1(a)c_1(a)g_1(a)\bar{\gamma}(t)-\mu_1(a)\bar{i}_1(t,a)-\mu_2(a)\bar{i}_2(t,a)\right)v_1^0(a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{s}_2(t,a) = -\left[\bar{\gamma}_2(t)\beta_2(a)c_2(a)g_2(a)-\mu_1(a)\bar{i}_1(t,a)-\mu_2(a)\bar{i}_2(t,a)\right]s_2^0(a) \\
\qquad -\left(b(a)+\psi_2(a)\right)\bar{s}_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{l}_2(t,a) = -\left(b(a)+k_2\right)\bar{l}_2(t,a) \\
\qquad +\bar{\gamma}_2(t)\beta_2(a)c_2(a)g_2(a)\left[(1-\phi_2)s_2^0(a)+\delta_2 v_2^0(a)\right] \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{i}_2(t,a) = k_2\bar{l}_2(t,a)+\phi_2\beta_2(a)c_2(a)g_2(a)\bar{\gamma}_2(t)s_2^0(a) \\
\qquad -\left(r_2+\mu_2(a)+b(a)\right)\bar{i}_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{j}_2(t,a) = r_2\bar{i}_2(t,a)-b(a)\bar{j}_2(t,a) \\
\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{v}_2(t,a) = \psi_2(a)\bar{s}_2(t,a)+\rho_1\bar{v}_1(t,a)-\left(\rho_2+b(a)\right)\bar{v}_2(t,a) \\
\qquad -\left(\delta_2\beta_2(a)c_2(a)g_2(a)\bar{\gamma}_2(t)-\mu_1(a)\bar{i}_1(t,a)-\mu_2(a)\bar{i}_2(t,a)\right)v_2^0(a) \\
\bar{\gamma}_i(t) = \int_0^{a_+}\hat{\beta}_i(a)\bar{i}_i(t,a)\mathrm{d}a
\end{cases} \tag{16}$$

with boundary conditions:

$$\bar{s}_i(t,0)=\bar{l}_i(t,0)=\bar{i}_i(t,0)=\bar{j}_i(t,0)=\bar{v}_i(t,0)=0,$$

we consider the exponential solutions of system (16) of the form:

$$\begin{cases}
\bar{s}_i(t,a)=\bar{s}_i(a)\mathrm{e}^{\lambda t};\ \bar{l}_i(t,a)=\bar{l}_i(a)\mathrm{e}^{\lambda t} \\
\bar{v}_i(t,a)=\bar{v}_i(a)\mathrm{e}^{\lambda t} \\
\bar{i}_i(t,a)=\bar{i}_i(a)\mathrm{e}^{\lambda t};\ \bar{j}_i(t,a)=\bar{j}_i(a)\mathrm{e}^{\lambda t}
\end{cases}. \tag{17}$$

The system (16) becomes:

$$
\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{s}_1(a) = -\big(b(a)+\psi_1(a)+\lambda\big)\bar{s}_1(a) - \big[\bar{\Gamma}_1\beta_1(a)c_1(a)g_1(a) - \mu_1(a)\bar{i}_1(a) - \mu_2(a)\bar{i}_2(a)\big]s_1^0(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{l}_1(a) = -\big(b(a)+k_1+\lambda\big)\bar{l}_1(a) + \bar{\Gamma}_1\beta_1(a)c_1(a)g_1(a)\big[(1-\phi_1)s_1^0(a) + \delta_1 v_1^0(a)\big] \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{i}_1(a) = \phi_1\beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1 + k_1\bar{l}_1(a) - \big(r_1+\mu_1(a)+b(a)+\lambda\big)\bar{i}_1(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{j}_1(a) = r_1\bar{i}_1(a) - \big(b(a)+\lambda\big)\bar{j}_1(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{v}_1(a) = -\big(\rho_1+b(a)+\lambda\big)\bar{v}_1(a) - \big(\delta_1\bar{\Gamma}\beta_1(a)c_1(a)g_1(a) - \mu_1(a)\bar{i}_1(a) - \mu_2(a)\bar{i}_2(a)\big)v_1^0(a) + \psi_1(a)\bar{s}(a) + \rho_2\bar{v}_2(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{s}_2(a) = -\big(b(a)+\psi_2(a)+\lambda\big)\bar{s}_2(a) - \big[\bar{\Gamma}_2\beta_2(a)c_2(a)g_2(a) - \mu_1(a)\bar{i}_1(a) - \mu_2(a)\bar{i}_2(a)\big]s_2^0(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{l}_2(a) = -\big(b(a)+k_2+\lambda\big)\bar{l}_2(a) + \bar{\Gamma}_2\beta_2(a)c_2(a)g_2(a)\big[(1-\phi_2)s_2^0(a) + \delta_2 v_2^0(a)\big] \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{i}_2(a) = \phi_2\beta_2(a)c_2(a)g_2(a)\bar{\Gamma}_2 + k_2\bar{l}_2(a) - \big(r_2+\mu_2(a)+b(a)+\lambda\big)\bar{i}_2(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{j}_2(a) = r_2\bar{i}_2(a) - \big(b(a)+\lambda\big)\bar{j}_2(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\bar{v}_2(a) = -\big(\rho_2+b(a)+\lambda\big)\bar{v}_2(a) - \big(\delta_2\bar{\Gamma}\beta_2(a)c_2(a)g_2(a) - \mu_1(a)\bar{i}_1(a) - \mu_2(a)\bar{i}_2(a)\big)v_2^0(a) + \psi_2(a)\bar{s}_2(a) + \rho_1\bar{v}_1(a) \\
\bar{\Gamma}_i = \int_0^{a_+}\hat{\beta}_i(a)\bar{i}_i(a)\mathrm{d}a
\end{cases}
\tag{18}
$$

with boundary conditions:

$$\bar{s}_i(0) = \bar{l}_i(0) = \bar{i}_i(0) = \bar{j}_i(0) = \bar{v}_i(0) = 0$$

Let

$$N_{\psi_i}(a) = (1-\phi_i)s_i^0(a) + \delta_i v_i^0(a). \tag{19}$$

From Equation (18), we obtain:

$$\bar{l}_i(a) = \bar{\Gamma}_i\int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+k_i+\lambda)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)N_{\psi_i}(\eta)\mathrm{d}\eta \tag{20}$$

$$\bar{i}_i(a) = \int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+r_i+\lambda+\mu_i(a))\mathrm{d}\tau}\big(k_i\bar{l}_i(\eta) + \bar{\Gamma}_i\phi_i\beta_i(\eta)c_i(\eta)g_i(\eta)\big)\mathrm{d}\eta. \tag{21}$$

Hence, by Equations ((20) and (21)) after changing order of integration, we obtain:

$$\bar{i}_i(a) = \bar{\Gamma}_i\int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+\mu_i(\tau)+r_i+\lambda)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta)\int_\eta^a \mathrm{e}^{-\int_\alpha^\eta (\mu_i(\tau)+r_i-k_i)\mathrm{d}\tau}\mathrm{d}\alpha\right]\mathrm{d}\eta. \tag{22}$$

Injecting (22) in the expression of $\bar{\Gamma}_i$, and dividing both sides the expression by $\bar{\Gamma}_i$ (since $\bar{\Gamma}_i \neq 0$), we get the characteristic equation:

$$1 = \int_0^{a_+}\hat{\beta}_i(a)\int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+\mu_i(\tau)+r_i+\lambda)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta)\int_\eta^a \mathrm{e}^{-\int_\alpha^\eta (\mu_i(\tau)+r_i-k_i)\mathrm{d}\tau}\mathrm{d}\alpha\right]\mathrm{d}\eta\mathrm{d}a. \tag{23}$$

Denote the right-hand side of Equation (23) by $G(\lambda)$ *i.e.*:

$$G_i(\lambda) = \int_0^{a_+}\hat{\beta}_i(a)\int_0^a \mathrm{e}^{-\int_\eta^a (b(\tau)+\mu_i(\tau)+r_i+\lambda)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i s_i^0(\eta) + k_i N_{\psi_i}(\eta)\int_\eta^a \mathrm{e}^{-\int_\alpha^\eta (\mu_i(\tau)+r_i-k_i)\mathrm{d}\tau}d\alpha\right]\mathrm{d}\eta\mathrm{d}a. \tag{24}$$

We define the net reproductive number as $\Re^i(\psi) = G_i(0)$, *i.e.*

$$\Re^i(\psi_i)=\int_0^{a_+}\hat{\beta}_i(a)\int_0^a e^{-\int_\eta^a(b(\tau)+\mu_i(\tau)+r_i)d\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i s_i^0(\eta)+k_i N_{\psi_i}(\eta)\int_\eta^a e^{-\int_\alpha^\eta(\mu_i(\tau)+r_i-k_i)d\tau}d\alpha\right]d\eta da \text{ . (25)}$$

We can obtain an expression for $\Re_0^i$ in a similar way as the derivation of $\Re^i(\psi)$ by considering Equation (1) without vaccination; *i.e.*, by assuming that $\psi_i(a)\equiv 0$ and neglecting the equation of vaccinated. It can be shown that $\Re_0^i=\Re^i(0)$ which is called the basic reproductive number (when a purely susceptible population is considered) (see [8]).

$$\Re_0^i=\Lambda_i\int_0^{a_+}\hat{\beta}_i(a)\int_0^a e^{-\int_\eta^a(b(\tau)+\mu_i(\tau)+r_i)d\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i+k_i(1-\phi_i)\int_\eta^a e^{-\int_\alpha^\eta(\mu_i(\tau)+r_i-k_i)d\tau}d\alpha\right]d\eta da \quad (26)$$

Let

$$\Re(\psi)=\max_i\Re^i(\psi_i) \text{ and } \Re_0=\max_i\Re_0^i.$$

4.3. Local Stability of the Disease-Free Equilibrium

Theorem 2. *The infection-free steady-state* (5) *is locally asymptotically stable* (*l.a.s.*) *if* $\Re(\psi)<1$ *and unstable if* $\Re(\psi)>1$.

Proof. Noticing that

$$G_i'(\lambda)<0;\ \lim_{\lambda\to+\infty}G_i(\lambda)=0;\ \lim_{\lambda\to-\infty}G_i(\lambda)=+\infty.$$

We know that Equation (23) has a unique negative real solution λ^* if, and only if, $G_i(0)<1$, hence, $\Re^i(\psi_i)<1$ (Also, Equation (23) has a unique positive (zero) real solution if $\Re^i(\psi_i)>1$ ($\Re^i(\psi_i)=1$). To show that λ^* is the dominant real part of roots of $G_i(\lambda)$, we let $\lambda=x+iy$ be an arbitrary complex solution to Equation (23). Note that

$$1=G_i(\lambda)=\left|G_i(x+iy)\right|\le G_i(x),$$

indicating that $R_e(\lambda)\le\lambda^*$. It follows that the infection-free steady state is l.a.s. if $\Re(\psi)<1$, and unstable if $\Re(\psi)>1$. □

In this corollary, we have the three cases of the unstability of the disease free equilibrium.

Corollary 1. 1) *whenever* $\Re^1(\psi_1)<1$ *and* $\Re^2(\psi_2)>1$, *the disease free is locally asymptotically stable in the first patch and unstable in the second.*

2) whenever $\Re^1(\psi_1)>1$ and $\Re^2(\psi_2)<1$, the disease free is unstable in the first patch and locally asymptotically stable in the second.

3) whenever $\Re^1(\psi_1)>1$ and $\Re^2(\psi_2)>1$, the disease free is unstable in the two patches.

4.4. Global Stability of the Disease-Free Equilibrium

Since $\mu_i(a)$ and $i_i(t,a)$ are bounded, there exists a positive constant R_c that satisfies

$$0\le\int_\eta^a\sum_{i=1}^2\mu_i(\tau)i_i(t-a+\tau,\tau)d\tau\le R_c \quad (*).$$

Corollary 2. *Assume that* $r_i+\mu_i(\tau)\le k_i$, *then we have*

$$\left[\mathrm{e}^{-\int_\eta^a (r_i+\mu_i(\tau)+b(\tau))\mathrm{d}\tau}\phi_i s_i(t-a+\eta,\eta)+k_i\left(\sigma_i j_i(t-a+\eta,\eta)+\delta_i v_i(t-\eta+a,\eta)\right)\right.$$
$$\left.+(1-\phi_i)s_i(t-a+\eta,\eta)\mathrm{e}^{-\int_\eta^a (k_i+b(\tau))\mathrm{d}\tau}\int_\eta^a \mathrm{e}^{-\int_\xi^a (r_i\mu_i(\tau)-k_i)\mathrm{d}\tau}\mathrm{d}\xi\right]$$
$$\le \Lambda_i \mathrm{e}^{-\int_\eta^a (b(\tau)+\mu_i(\tau)+r_i)\mathrm{d}\tau}\left[\phi_i+k_i(1-\phi_i)\int_\eta^a \mathrm{e}^{-\int_\xi^\eta (\mu_i(\tau)+r_i-k_i)\mathrm{d}\tau}\mathrm{d}\xi\right].$$

Theorem 3. *The disease-free equilibrium of system* (5) *is globally asymptotically stable if* $\Re_0<1$ *and* $R_c<\ln\left(\frac{1}{\Re_0}\right)$.

Proof. The proof consist to show that

$i_i(t,a)\to 0$; $j_i(t,a)\to 0$; $l_i(t,a)\to 0$;

$s_i(t,a)\to s_i^0(a)$ and $v_i(t,a)\to \Lambda_i - s_i^0(a)$, when $t\to+\infty$.

Integrating system (5) along characteristic lines we get

$$l_i(t,a)=\int_0^a \mathrm{e}^{-\int_\eta^a \left(b(\tau)-\sum_{i=1}^2 \mu_i(\tau)i_i(t-a+\tau,\tau)\mathrm{d}\tau+k_i\right)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\lambda_i(t-a+\eta)$$
$$\times\left[\sigma_i j_i(t-a+\eta,\eta)+\delta_i v_i(t-\eta+a,\eta)+(1-\phi_i)s_i(t-a+\eta,\eta)\right]\mathrm{d}\eta,\ a<t \tag{27}$$

$$i_i(t,a)=\int_0^a \mathrm{e}^{-\int_\xi^a \left(b(\tau)-\sum_{i=1}^2 \mu_i(\tau)i_i(t-a+\tau,\tau)+r_i+\mu_i(\tau)\right)\mathrm{d}\tau}$$
$$\times\left[\phi_i\beta_i(\xi)c_i(\xi)g_i(\xi)\lambda_i(t-a+\xi)+k_i l(t-a+\xi,\xi)\right]\mathrm{d}\xi,\ a<t \tag{28}$$

Injecting (27) in (28), and changing order of integration, we obtain:

$$i_i(t,a)=\int_0^a \mathrm{e}^{\int_\eta^a \sum_{i=1}^2 \mu_i(\tau)i_i(t-a+\tau,\tau)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\lambda_i(t-a+\eta)$$
$$\times\left[\mathrm{e}^{-\int_\eta^a (r_i+\mu_i(\tau)+b(\tau))\mathrm{d}\tau}\phi_i s_i(t-a+\eta,\eta)+k_i\left(\sigma_i j_i(t-a+\eta,\eta)+\delta_i v_i(t-\eta+a,\eta)\right)\right.$$
$$\left.+(1-\phi_i)s_i(t-a+\eta,\eta)\mathrm{e}^{-\int_\eta^a (k_i+b(\tau))\mathrm{d}\tau}\int_\eta^a \mathrm{e}^{-\int_\xi^\eta (r_i+\mu_i(\tau)-k_i)\mathrm{d}\tau}\mathrm{d}\xi\right)\right]\mathrm{d}\eta \tag{29}$$

Injecting (29) in $\lambda_i(t)$, and changing order of integration, we obtain:

$$\lambda_i(t)=\int_0^{a_+}\hat{\beta}_i(a)\int_0^a \mathrm{e}^{\int_\eta^a \sum_{i=1}^2 \mu_i(\tau)i_i(t-a+\tau,\tau)\mathrm{d}\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\lambda_i(t-a+\eta)$$
$$\times\left[\mathrm{e}^{-\int_\eta^a (r_i+\mu_i(\tau)+b(\tau))\mathrm{d}\tau}\phi_i s_i(t-a+\eta,\eta)+k_i\left(\sigma_i j_i(t-a+\eta,\eta)+\delta_i v_i(t-\eta+a,\eta)\right)\right.$$
$$\left.+(1-\phi_i)s_i(t-a+\eta,\eta)\mathrm{e}^{-\int_\eta^a (k_i+b(\tau))\mathrm{d}\tau}\int_\eta^a \mathrm{e}^{-\int_\xi^\eta (r_i+\mu_i(\tau)-k_i)\mathrm{d}\tau}\mathrm{d}\xi\right)\right]\mathrm{d}\eta\mathrm{d}a \tag{30}$$

By using corollary 2, inequality (*) and Fatou's lemma, we have

$$\lim_{t\to+\infty}\lambda_i(t)\le \mathrm{e}^{R_c}\Re_0^i \limsup_{t\to+\infty}\lambda_i(t).$$

Since $\mathrm{e}^{R_c}\Re_0^i<1$, $\Rightarrow \limsup_{t\to+\infty}\lambda_i(t)=0$ $\Rightarrow$

$$\begin{cases}\lim\limits_{t\to+\infty} i_i(t,a)=\lim\limits_{t\to+\infty} j_i(t,a)=\lim\limits_{t\to+\infty} l_i(t,a)=0\\ \lim\limits_{t\to+\infty} s_i(t,a)=s_i^0(a),\ \lim\limits_{t\to+\infty} v_i(t,a)=\Lambda_i-s_i^0(a).\end{cases}$$

□

Corollary 3. *The disease-free equilibrium is globally asymptotically in*:

1) the first sub-population if $\Re_0^1 < 1$ and $0 \le \int_\eta^a \mu_1(\tau) i_1(t-a+\tau,\tau)\mathrm{d}\tau < \ln\left(\frac{1}{\Re_0^1}\right)$.

2) the second sub-population if $\Re_0^2 < 1$ and $0 \le \int_\eta^a \mu_2(\tau) i_2(t-a+\tau,\tau)\mathrm{d}\tau < \ln\left(\frac{1}{\Re_0^2}\right)$.

For this disease can disappear without any form of intervention, according to these results we must ensure that there is no new infected and the infectious rate does not reach a certain spread.

5. Existence of an Endemic State

There exists three endemic steady state of system (5) whenever $\Re(\psi) > 1$.

5.1. The First Boundary Endemic Equilibrium

Theorem 4. *A boundary endemic equilibrium of the form* $\bar{E}_1^* = \left(\bar{s}_1^*(a), \bar{l}_1^*(a), \bar{i}_1^*(a), \bar{j}_1^*(a), \bar{v}_1^*(a), \bar{s}_2^*(a), 0, 0, 0, \bar{v}_2^*(a)\right)$ *whenever* $\Re^1(\psi_1) > 1$ *and* $\Re^2(\psi_2) < 1$. *This means that the disease is endemic in the first sub-population and dies out in the second sub-population.*

Proof. The method commonly used to find an endemic steady state for age-structure models consists of obtaining explicit expressions for a time independent solution of system (5)

$\bar{E}_1^* = \left(\bar{s}_1^*(a), \bar{l}_1^*(a), \bar{i}_1^*(a), \bar{j}_1^*(a), \bar{v}_1^*(a), \bar{s}_2^*(a), 0, 0, 0, \bar{v}_2^*(a)\right)$ satisfies the following equations:

$$\begin{cases}
\frac{\mathrm{d}}{\mathrm{d}a}\bar{s}_1^*(a) = b_1(a) - \left[\beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1^* + \psi_1(a) + b(a) - \mu_1(a)\bar{i}_1^*(a)\right]s_1^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{l}_1^*(a) = \beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1^*\left[(1-\phi_1)\bar{s}_1^*(a) + \delta_1\bar{v}_1^*(a) + \sigma_1\bar{j}_1^*(a)\right] - \left(b(a) + k_1 - \mu_1(a)\bar{i}_1^*(a)\right)\bar{l}_1^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{i}_1^*(a) = k_1\bar{l}_1^*(a) + \phi_1\beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1^*\bar{s}_1^*(a) - \left[r_1 + b(a) + \mu_1(a) - \mu_1(a)\bar{i}_1^*(a)\right]\bar{i}_1^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{j}_1^*(a) = r_1\bar{i}_1^*(a) - \left(\sigma_1\beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1^* + b(a) - \mu_1(a)\bar{i}_1^*(a)\right)\bar{j}_1^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{v}_1^*(a) = \psi_1(a)\bar{s}_1^*(a) + \rho_2\bar{v}_2^*(a) - \left(\delta_1\beta_1(a)c_1(a)g_1(a)\bar{\Gamma}_1^* + b(a) - \mu_1(a)\bar{i}_1^*(a)\right)v_1^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{s}_2^*(a) = b_2(a) - \left[\psi_2(a) + b(a) - \mu_1(a)\bar{i}_1^*(a)\right]s_2^*(a) \\
\frac{\mathrm{d}}{\mathrm{d}a}\bar{v}_2^*(a) = \psi_2(a)\bar{s}_2^*(a) + \rho_1\bar{v}_1^*(a) - \left(b(a) - \mu_1(a)\bar{i}_1^*(a)\right)\bar{v}_2^*(a) \\
\bar{\Gamma}_1^* = \int_0^{a_+}\hat{\beta}_1(a)\bar{i}_1^*(a)\mathrm{d}a
\end{cases} \tag{31}$$

with the initial conditions:

$$\bar{s}_i^*(0) = \Lambda_i;\ \bar{i}_1^*(0) = \bar{l}_1^*(0) = \bar{v}_i^*(0) = \bar{j}_1^*(0) = 0.$$

Let

$$h_1\left(\eta, \bar{\Gamma}_1^*\right) = (1-\phi_1)\bar{s}_1^*(\eta) + \delta_1\bar{v}_1^*(\eta) + \sigma_1\bar{j}_1^*(\eta). \tag{32}$$

Integrating system (31), we obtain:

$$\begin{aligned}\bar{s}_1^*(a) &= \Lambda_1 \mathrm{e}^{-\int_0^a \left[\bar{\Gamma}_1^* \beta_1(\tau) c_1(\tau) g_1(\tau) - \mu_1(\tau) \bar{i}_1^*(\eta) + b(\eta) + \psi_1(\eta)\right] \mathrm{d}\eta} \\ &\quad + \int_0^a b(\eta) \mathrm{e}^{-\int_\eta^a \left[\bar{\Gamma}_1^* (1-\phi_1) \beta_1(\tau) c_1(\tau) g_1(\tau) - \mu_1(\tau) \bar{i}_1^*(\eta) + b(\eta) + \psi_1(\eta)\right] \mathrm{d}\eta} \mathrm{d}\eta\end{aligned} \tag{33}$$

$$\bar{i}_1^*(a) = \int_0^a \left[k_1 \bar{l}_1^*(\eta) + \phi_1 \beta_1(\eta) c_1(\eta) g_1(\eta) \bar{\Gamma}_1^* \bar{s}_1^*(\eta)\right] \mathrm{e}^{-\int_\eta^a \left(r_1 - \mu_1(\eta) \bar{i}_1^*(\tau) + b(\tau) + \mu_1(\tau)\right) \mathrm{d}\tau} \mathrm{d}\eta \tag{34}$$

$$\bar{j}_1^*(a) = \int_0^a \bar{i}_1^*(\eta) \mathrm{e}^{-\int_\eta^a \left(\sigma_1 \beta_1(\tau) c_1(\tau) g_1(\tau) \bar{\Gamma}_1^* + b(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau)\right) \mathrm{d}\tau} \mathrm{d}\eta \tag{35}$$

$$\bar{v}_1^*(a) = \int_0^a \left[\psi_1(\eta) \bar{s}_1^*(\eta) + \rho_2 \bar{v}_2^*(\eta)\right] \mathrm{e}^{-\int_\eta^a \left(\delta_1 \bar{\Gamma}_1^* c_1(\tau) g_1(\tau) \beta_1(\tau) \bar{\Gamma}_1^* + b(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau)\right) \mathrm{d}\tau} \tag{36}$$

$$\bar{l}_1^*(a) = \bar{\Gamma}_1^* \int_0^a \mathrm{e}^{-\int_\eta^a \left(b(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau) + k\right) \mathrm{d}\tau} \beta_1(\eta) c_1(\eta) g_1(\eta) \bar{s}_1^*(\eta) h_1\left(\eta, \bar{\Gamma}_1^*\right) \mathrm{d}\eta \tag{37}$$

$$\bar{s}_2^*(a) = \Lambda_2 \mathrm{e}^{-\int_0^a \left(\psi_2(\tau) + b(\tau) - \mu_1 \bar{i}_1^*(\tau)\right) \mathrm{d}\tau} + \int_0^a b(\eta) \mathrm{e}^{-\int_\eta^a \left(\psi_2(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau)\right) \mathrm{d}\tau} \mathrm{d}\eta \tag{38}$$

$$\bar{v}_2^*(a) = \int_0^a \left[\psi_2(\eta) \bar{s}_2^*(\eta) + \rho_1 \bar{v}_1^*(\eta)\right] \mathrm{e}^{-\int_\eta^a \left(b(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau)\right) \mathrm{d}\tau} \mathrm{d}\eta. \tag{39}$$

By injecting (37) in (34), we obtain:

$$\begin{aligned}\bar{i}_1^*(a) &= \bar{\Gamma}_1^* \int_0^a \beta_1(\eta) c_1(\eta) g_1(\eta) \mathrm{e}^{-\int_\eta^a \left(\beta(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau) + r_1 + \mu_1(\tau)\right) \mathrm{d}\tau} \\ &\quad \times \left[\phi_1 \bar{s}_1^*(\eta) + k_1 h_1\left(\eta, \bar{\Gamma}_1^*\right) \int_\eta^a \mathrm{e}^{-\int_\xi^\eta (r_1 + \mu_1(\tau) - k_1) \mathrm{d}\tau} \mathrm{d}\xi\right] \mathrm{d}\eta\end{aligned}. \tag{40}$$

Injecting (40) in the expression of $\bar{\Gamma}_1^*$, and dividing by $\bar{\Gamma}_1^*$ (since $\bar{\Gamma}_1^* \neq 0$)we obtain:

$$\begin{aligned}1 &= \int_0^{a_+} \hat{\beta}_1(a) \int_0^a \beta_1(\eta) c_1(\eta) g_1(\eta) \mathrm{e}^{-\int_\eta^a \left(\beta(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau) + r_1 + \mu_1(\tau)\right) \mathrm{d}\tau} \\ &\quad \times \left[\phi_1 \bar{s}_1^*(\eta) + k_1 h_1\left(\eta, \bar{\Gamma}_1^*\right) \int_\eta^a \mathrm{e}^{-\int_\xi^\eta (r_1 + \mu_1(\tau) - k_1) \mathrm{d}\tau} \mathrm{d}\xi\right] \mathrm{d}\eta \mathrm{d}a\end{aligned}. \tag{41}$$

Let H_1, the function define by:

$$\begin{aligned}H_1\left(\bar{\Gamma}_1^*\right) &= \int_0^{a_+} \hat{\beta}_1(a) \int_0^a \beta_1(\eta) c_1(\eta) g_1(\eta) e^{-\int_\eta^a \left(\beta(\tau) - \mu_1(\tau) \bar{i}_1^*(\tau) + r_1 + \mu_1(\tau)\right) \mathrm{d}\tau} \\ &\quad \times \left[\phi_1 \bar{s}_1^*(\eta) + k_1 h_1\left(\eta, \bar{\Gamma}_1^*\right) \int_\eta^a \mathrm{e}^{-\int_\xi^\eta (r_1 + \mu_1(\tau) - k_1) \mathrm{d}\tau} \mathrm{d}\xi\right] \mathrm{d}\eta \mathrm{d}a\end{aligned}. \tag{42}$$

Since $h_1(\eta, 0) = N_{\psi_1}(\eta)$ *i.e.* when $\bar{\Gamma}_1^* = 0, \bar{s}_i^*(a) = s_i^0(a)$ and $\bar{v}_i^*(a) = v_i^0(a)$, so the net reproductive number is given by

$$H_1(0) = \Re^1(\psi_1)$$

i.e.

$$\Re^1(\psi_1) = \int_0^{a_+} \hat{\beta}_1(a) \int_0^a \mathrm{e}^{-\int_\eta^a \left(b(\tau) + \mu_1(\tau) + r_1\right) \mathrm{d}\tau} \beta_1(\eta) c_1(\eta) g_1(\eta) \left[\phi_1 s_1^0(\eta) + k_1 N_{\psi_1}(\eta) \int_\eta^a \mathrm{e}^{-\int_\alpha^\eta (\mu_1(\tau) + r_1 - k_1) \mathrm{d}\tau} \mathrm{d}\alpha\right] \mathrm{d}\eta \mathrm{d}a.$$

We now see that an endemic steady state exists if Equation (41) has a positive solution.

Since $H_1(0) = \Re_1(\psi_1)$, hence $H_1(0) > 1$. We know that $\bar{s}_1^*(a) + \bar{l}_1^*(a) + \bar{i}_1^*(a) + \bar{v}_1^*(a) + \bar{j}_1^*(a) = \Lambda_1 < 1$. Hence

$$\bar{i}_1^*(a) < 1. \tag{43}$$

Since $\overline{\Gamma}_1^* > 0$, from (42) and (43) we obtain:

$$\begin{aligned}\overline{\Gamma}_1^* H_1\left(\overline{\Gamma}_1^*\right) &= \int_0^{a_+} \hat{\beta}_1(a) \int_0^a \overline{\Gamma}_1^* \beta_1(\eta) c_1(\eta) g_1(\eta) \mathrm{e}^{-\int_\eta^a \left(\beta(\tau)-\mu_1(\tau)\overline{i}_1^*(\tau)+r_1+\mu_1(\tau)\right)\mathrm{d}\tau} \\ &\quad \times \left[\phi_1 \overline{s}_1^*(\eta) + k_1 h_1\left(\eta, \overline{\Gamma}_1^*\right) \int_\eta^a \mathrm{e}^{-\int_\xi^\eta \left(r_1+\mu_1(\tau)-k_1\right)\mathrm{d}\tau} \mathrm{d}\xi\right] \mathrm{d}\eta \mathrm{d}a \\ &< \int_0^{a_+} \hat{\beta}_1(a)\mathrm{d}a = \beta_1^+.\end{aligned}$$

In particular, for $\overline{\Gamma}_1^* = \beta_1^+$, we have $H\left(\beta_1^+\right) < 1$, but $H(0) > 1$. Since H_1 is continous function of $\overline{\Gamma}_1^*$, we conclude that $H\left(\overline{\Gamma}_1^*\right) = 1$, has a positive solution $\hat{\Gamma}_1^*$ on $]0;\beta^+[$. This solution may not be unique since H may not be monotone ($H\left(\overline{\Gamma}_1^*\right)$ depends on $h_1\left(\eta, \overline{\Gamma}_1^*\right)$ which is defined implicitly). It follows that when $\Re^1(\psi_1) > 1$, there exists an endemic steady state distribution which is given by the unique solution of Equation (41) corresponding to $\hat{\Gamma}_1^*$. □

5.2. The Second Boundary Endemic Equilibrium

Theorem 5. *A boundary endemic equilibrium of the form* $\overline{E}_2^{**} = \left(\overline{s}_1^{**}(a), 0, 0, \overline{v}_1^{**}(a), \overline{s}_2^{**}(a), \overline{l}_2^{**}(a), \overline{i}_2^{**}(a), \overline{j}_2^{**}(a), \overline{v}_2^{**}(a)\right)$ *whenever* $\Re^1(\psi_1) < 1$ *and* $\Re^2(\psi_2) > 1$. *This means that the disease is dies out in the first sub-population and is endemic in the second sub-population.*

Proof. (Ideas of proof)

$\overline{E}_2^{**} = \left(\overline{s}_1^{**}(a), 0, 0, \overline{v}_1^{**}(a), \overline{s}_2^{**}(a), \overline{l}_2^{**}(a), \overline{i}_2^{**}(a), \overline{j}_2^{**}(a), \overline{v}_2^{**}(a)\right)$ satisfies the following equations:

$$\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{s}_1^{**}(a) = b_1(a) - \left[\psi_1(a) + b(a) - \mu_2(a)\overline{i}_2^{**}(a)\right]\overline{s}_1^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{v}_1^{**}(a) = \psi_1(a)\overline{s}_1^{**}(a) + \rho_2\overline{v}_2^{**}(a) - \left(b(a) - \mu_2(a)\overline{i}_2^{**}(a)\right)\overline{v}_1^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{s}_2^{**}(a) = b_2(a) - \left[\beta_2(a)c_2(a)g_2(a)\overline{\Gamma}_2^{**} + \psi_2(a) + b(a) - \mu_2(a)\overline{i}_2^{**}(a)\right]s_2^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{l}_2^{**}(a) = \beta_2(a)c_2(a)g_2(a)\overline{\Gamma}_2^{**}\left[(1-\phi_2)\overline{s}_2^{**}(a) + \delta_2\overline{v}_2^{**}(a) + \sigma_2\overline{j}_2^{**}(a)\right] - \left(b(a) + k_2 - \mu_2(a)\overline{i}_2^{**}(a)\right)\overline{l}_2^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{i}_2^{**}(a) = k_2\overline{l}_2^{**}(a) + \phi_2\beta_2(a)c_2(a)g_2(a)\overline{\Gamma}_2^*\overline{s}_2^*(a) - \left[r_2 + b(a) + \mu_2(a) - \mu_2(a)\overline{i}_2^{**}(a)\right]\overline{i}_2^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{j}_2^{**}(a) = r_2\overline{i}_2^{**}(a) - \left(\sigma_2\beta_2(a)c_2(a)g_2(a)\overline{\Gamma}_1^{**} + b(a) - \mu_2(a)\overline{i}_2^{**}(a)\right)\overline{j}_2^{**}(a) \\
\dfrac{\mathrm{d}}{\mathrm{d}a}\overline{v}_2^{**}(a) = \psi_2(a)\overline{s}_2^{**}(a) + \rho_2\overline{v}_2^{**}(a) - \left(\delta_2\beta_2(a)c_2(a)g_2(a)\overline{\Gamma}_2^{**} + b(a) - \mu_2(a)\overline{i}_2^{**}(a)\right)v_2^{**}(a) \\
\overline{\Gamma}_2^{**} = \int_0^{a_+}\hat{\beta}_2(a)\overline{i}_2^{**}(a)\mathrm{d}a
\end{cases} \tag{44}$$

with the initial conditions:

$$\overline{s}_i^{**}(0) = \Lambda_i;\ \overline{i}_2^{**}(0) = \overline{l}_2^{**}(0) = \overline{v}_i^{**}(0) = \overline{j}_2^{**}(0) = 0. \tag{45}$$

Let

$$h_2\left(\eta, \overline{\Gamma}_2^{**}\right) = (1-\phi_2)\overline{s}_2^{**}(\eta) + \delta_2\overline{v}_2^{**}(\eta) + \sigma_2\overline{j}_2^{**}(\eta). \tag{46}$$

Integrating system (51), we obtain:

$$\overline{s}_1^{**}(a) = \Lambda_1 \mathrm{e}^{-\int_0^a \left(\psi_1(\tau)+b(\tau)-\mu_2\overline{i}_2^*(\tau)\right)\mathrm{d}\tau} + \int_0^a b(\eta)\mathrm{e}^{-\int_\eta^a \left(\psi_1(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\tau)\right)\mathrm{d}\tau}\mathrm{d}\eta \tag{47}$$

$$\overline{v}_1^{**}(a)=\int_0^a\left[\psi_2(\eta)\overline{s}_2^{**}(\eta)+\rho_1\overline{v}_1^{**}(\eta)\right]\mathrm{e}^{-\int_\eta^a\left(b(\tau)-\mu_1(\tau)\overline{i}_1^{**}(\tau)\right)\mathrm{d}\tau}\mathrm{d}\eta \tag{48}$$

$$\begin{aligned}\overline{s}_2^{**}(a)=&\Lambda_2\mathrm{e}^{-\int_0^a\left[\overline{\Gamma}_2^*\beta_2(\tau)c_2(\tau)g_2(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\eta)+b(\eta)+\psi_2(\eta)\right]\mathrm{d}\eta}\\&+\int_0^a b(\eta)\mathrm{e}^{-\int_\eta^a\left[\overline{\Gamma}_2^{**}(1-\phi_2)\beta_2(\tau)c_2(\tau)g_2(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\tau)+b(\tau)+\psi_2(\tau)\right]\mathrm{d}\tau}\mathrm{d}\eta\end{aligned} \tag{49}$$

$$\overline{i}_2^{**}(a)=\int_0^a\left[k_2\overline{l}_2^{**}(\eta)+\phi_2\beta_2(\eta)c_2(\eta)g_2(\eta)\overline{\Gamma}_2^*\overline{s}_2^{**}(\eta)\right]\mathrm{e}^{-\int_\eta^a\left(r_1-\mu_1(\eta)\overline{i}_2^{**}(\tau)+b(\tau)+\mu_2(\tau)\right)\mathrm{d}\tau}\mathrm{d}\eta \tag{50}$$

$$\overline{j}_2^{**}(a)=\int_0^a\overline{i}_2^{**}(\eta)\mathrm{e}^{-\int_\eta^a\left(\sigma_2\beta_2(\tau)c_2(\tau)g_2(\tau)\overline{\Gamma}_2^{**}+b(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\tau)\right)\mathrm{d}\tau}\mathrm{d}\eta \tag{51}$$

$$\overline{v}_2^{**}(a)=\int_0^a\left[\psi_2(\eta)\overline{s}_2^{**}(\eta)+\rho_1\overline{v}_1^{**}(\eta)\right]\mathrm{e}^{-\int_\eta^a\left(\delta_2\overline{\Gamma}_2^{**}c_2(\tau)g_2(\tau)\beta_2(\tau)\overline{\Gamma}_2^{**}+b(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\tau)\right)\mathrm{d}\tau} \tag{52}$$

$$\overline{l}_2^{**}(a)=\overline{\Gamma}_2^*\int_0^a\mathrm{e}^{-\int_\eta^a\left(b(\tau)-\mu_2(\tau)\overline{i}_2^{**}(\tau)+k\right)\mathrm{d}\tau}\beta_2(\eta)c_2(\eta)g_2(\eta)\overline{s}_2^{**}(\eta)h_2\left(\eta,\overline{\Gamma}_2^*\right)\mathrm{d}\eta\,. \tag{53}$$

Hence, by the similar method using in theorem 4, we obtain the result. □

5.3. The Interior Endemic Equilibrium

Theorem 6. *An interior endemic equilibrium of the form*

$$E^*=\left(s_1^*(a),l_1^*(a),i_1^*(a),j_1^*(a),v_1^*(a),s_2^*(a),l_2^*(a),i_2^*(a),j_2^*(a),v_2^*(a)\right)$$

whenever $\Re^1(\psi_1)>1$ and $\Re^2(\psi_2)>1$, which corresponds to case when the disease persists in the two sub-populations.

Proof. $E^*=\left(s_1^*(a),l_1^*(a),i_1^*(a),j_1^*(a),v_1^*(a),s_2^*(a),l_2^*(a),i_2^*(a),j_2^*(a),v_2^*(a)\right)$ satisfies the following equations:

$$\begin{cases}\dfrac{\mathrm{d}}{\mathrm{d}a}s_1^*(a)=b_1(a)-\left[\beta_1(a)c_1(a)g_1(a)\Gamma_1^*+\psi_1(a)+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right]s_1^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}l_1^*(a)=\beta_1(a)c_1(a)g_1(a)\Gamma_1^*\left[(1-\phi_1)s_1^*(a)+\delta_1v_1^*(a)+\sigma_1j_1^*(a)\right]-\left(b(a)+k_1-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right)l_1^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}i_1^*(a)=k_1l_1^*(a)+\phi_1\beta_1(a)c_1(a)g_1(a)\Gamma_1^*s_1^*(a)-\left[r_1+b(a)+\mu_1(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right]i_1^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}j_1^*(a)=r_1i_1^*(a)-\left(\sigma_1\beta_1(a)c_1(a)g_1(a)\Gamma_1^*+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right)j_1^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}v_1^*(a)=\psi_1(a)s_1^*(a)+\rho_2v_2^*(a)-\left(\delta_1\beta_1(a)c_1(a)g_1(a)\Gamma_1^*+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)+\rho_1\right)v_1^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}s_2^*(a)=b_2(a)-\left[\beta_2(a)c_2(a)g_2(a)\Gamma_2+\psi_2(a)+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2(a)\right]s_2(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}l_2^*(a)=\beta_2(a)c_2(a)g_2(a)\Gamma_2^*\left[(1-\phi_2)s_2^*(a)+\delta_2v_2^*(a)+\sigma_2j_2^*(a)\right]-\left(b(a)+k_2-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right)l_2^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}i_2^*(a)=k_2l_2^*(a)+\phi_2\beta_2(a)c_2(a)g_2(a)\Gamma_2^*s_2^*(a)-\left[r_2+b(a)+\mu_2(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right]i_2^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}j_2(a)=r_2i_2^*(a)-\left(\sigma_2\beta_2(a)c_2(a)g_2(a)\Gamma_2^*+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)\right)j_2^*(a)\\ \dfrac{\mathrm{d}}{\mathrm{d}a}v_2^*(a)=\psi_2(a)s_2^*(a)+\rho_1v_1^*(a)-\left(\delta_2\beta_2(a)c_2(a)g_2(a)\Gamma_2^*+b(a)-\mu_1(a)i_1^*(a)-\mu_2(a)i_2^*(a)+\rho_2\right)v_2^*(a)\\ \Gamma_i^*=\int_0^{a_+}\hat{\beta}_i(a)i_i^*(a)\mathrm{d}a\end{cases} \tag{54}$$

with the initial conditions:

$$s_i^*(0)=\Lambda_i;\quad l_i^*(0)=i_i^*(0)=j_i^*(0)=v_i^*(0)=0 \tag{55}$$

$$\begin{aligned}s_i^*(a)=\Lambda_i e^{-\int_0^a\left[\beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma^*+b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)\right]d\tau}\\+\int_0^a b_i(\eta)e^{-\int_\eta^a\left[\beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma^*+b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)\right]d\tau}d\eta.\end{aligned} \tag{56}$$

Let

$$h_i^*(\eta,\Gamma_i^*)=(1-\phi_i)s_i^*(\eta)+\delta_i v_i^*(\eta)+\sigma_i j_i^*(\eta) \tag{57}$$

$$l_i^*(a)=\Gamma_i^*\int_0^a\beta_i(\eta)c_i(\eta)g_i(\eta)h_i(\eta,\Gamma_i^*)e^{-\int_\eta^a\left(b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)\right)d\tau}d\eta \tag{58}$$

$$i_i^*(a)=\int_0^a\left[k_i l_i^*(\eta)+\phi_i\beta_i(\eta)c_i(\eta)g_i(\eta)\Gamma_i^*s_i^*(\eta)\right]e^{-\int_\eta^a\left(b(\tau)+r_i+\mu_i(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2(\tau)\right)d\tau}d\eta \tag{59}$$

$$j_i^*(a)=r_i\int_0^a i_i^*(\eta)e^{-\int_\eta^a\left(\sigma_i\beta_i(\tau)c_i(\tau)g_i(\tau)\Gamma_i^*+b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\eta)\right)d\tau}d\eta \tag{60}$$

$$v_1^*(a)=\int_0^a\left(\psi_1(\eta)s_1^*(\eta)+\rho_2 v_2^*(\eta)\right)e^{-\int_\eta^a\left(\delta_1\beta_1(\tau)c_1(\tau)g_1(\tau)\Gamma_1^*+b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)\right)d\tau}d\eta \tag{61}$$

$$v_2^*(a)=\int_0^a\left(\psi_2(\eta)s^*(\eta)+\rho_1 v_1^*(\eta)\right)e^{-\int_\eta^a\left(\delta_2\beta_2(\tau)c_2(\tau)g_2(\tau)\Gamma^*+b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)\right)d\tau}d\eta. \tag{62}$$

By injecting (58) in (59), we obtain:

$$\begin{aligned}i_i^*(a)=\Gamma_i^*\int_0^a\beta_i(\eta)c_i(\eta)g_i(\eta)e^{-\int_\eta^a\left(b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)-r_i+\mu_i(\tau)\right)d\tau}\\\times\left[\phi_i s_i^*(\eta)+k_i h_i(\eta,\Gamma_i^*)\int_\eta^a e^{-\int_\xi^\eta(r_i+\mu_i(\tau)-k_i)d\tau}d\xi\right]d\eta.\end{aligned} \tag{63}$$

By injecting (63) in the expression of Γ_i^*, and dividing by Γ_i^* (since $\Gamma_i^*\neq 0$) we obtain:

$$\begin{aligned}1=\int_0^{a_+}\hat{\beta}_i(a)\int_0^a\beta_i(\eta)c_i(\eta)g_i(\eta)e^{-\int_\eta^a\left(b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)-r_i+\mu_i(\tau)\right)d\tau}\\\times\left[\phi_i s_i^*(\eta)+k_i h_i(\eta,\Gamma_i^*)\int_\eta^a e^{-\int_\xi^\eta(r_i+\mu_i(\tau)-k_i)d\tau}d\xi\right]d\eta da.\end{aligned} \tag{64}$$

Let H_i, the function define by:

$$\begin{aligned}H_i(\Gamma_i^*)=\int_0^{a_+}\hat{\beta}_i(a)\int_0^a\beta_i(\eta)c_i(\eta)g_i(\eta)e^{-\int_\eta^a\left(b(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)-r_i+\mu_i(\tau)\right)d\tau}\\\times\left[\phi_i s_i^*(\eta)+k_i h_i(\eta,\Gamma_i^*)\int_\eta^a e^{-\int_\xi^\eta(r_i+\mu_i(\tau)-k_i)d\tau}d\xi\right]d\eta da.\end{aligned} \tag{65}$$

Since $h_i(\eta,0)=N_{\psi_i}(\eta)$ *i.e.* when $\Gamma_i^*=0$, $s_i^*(a)=s_i^0(a)$ and $v_i^*(a)=v_i^0(a)$, so the net reproductive number is given by

$H_i(0)=\Re^i(\psi_i)$, *i.e.*

$$\Re^i(\psi_i)=\int_0^{a_+}\hat{\beta}_i(a)\int_0^a e^{-\int_\eta^a\left(b(\tau)+\mu_i(\tau)+r_i\right)d\tau}\beta_i(\eta)c_i(\eta)g_i(\eta)\left[\phi_i s_i^0(\eta)+k_i N_{\psi_i}(\eta)\int_\eta^a e^{-\int_\alpha^\eta(\mu_i(\tau)+r_i-k_i)d\tau}d\alpha\right]d\eta da.$$

We now see that an endemic steady state exists if Equation (64) has a positive solution. Since

$H_i(0)=\Re^i(\psi_i)$, hence $H_i(0)>1$. We know that

$s_i^*(a)+l_i^*(a)+i_i^*(a)+v_i^*(a)+j_i^*(a)=\Lambda_i<1$. Hence

$$i_i^*(a)<1 \tag{66}$$

Since $\Gamma_i^*>0$, from (65) and (66) we obtain:

$$\begin{aligned}\Gamma_i^* H_i\left(\Gamma_i^*\right) &= \int_0^{a_+}\hat{\beta}_i(a)\int_0^a \Gamma_i^*\beta_i(\eta)c_i(\eta)g_i(\eta)\mathrm{e}^{-\int_\eta^a\left(\beta(\tau)-\mu_1(\tau)i_1^*(\tau)-\mu_2(\tau)i_2^*(\tau)-r_i+\mu_i(\tau)\right)\mathrm{d}\tau}\\ &\quad\times\left[\phi_i s_i^*(\eta)+k_i h_i\left(\eta,\Gamma_i^*\right)\int_\eta^a \mathrm{e}^{-\int_\xi^\eta\left(r_i+\mu_i(\tau)-k_i\right)\mathrm{d}\tau}\mathrm{d}\xi\right]\mathrm{d}\eta\mathrm{d}a\\ &< \int_0^{a_+}\hat{\beta}_i(a)\mathrm{d}a=\beta_i^+.\end{aligned}$$

In particular, for $\Gamma_i^*=\beta_i^+$, we have $H_i\left(\beta_i^+\right)<1$, but $H_i(0)>1$. Since H_i is continous function of Γ_i^*, we conclude that $H\left(\Gamma_i^*\right)=1$, has a positive solution $\hat{\Gamma}_i^*$ on $]0;\beta_i^+[$. This solution may not be unique since H_i may not be monotone ($H\left(\Gamma_i^*\right)$ depends on $h\left(\eta,\Gamma_i^*\right)$ which is defined implicitly). It follows that when $\Re^i\left(\psi_i\right)>1$, there exists an endemic steady state distribution which is given by the unique solution of Equation (64) corresponding to $\hat{\Gamma}_i^*$. □

5.4. Simulation

In this section, when $\Re^1\left(\psi_1\right)>1$ and $\Re^2\left(\psi_2\right)>1$ we will evaluate the impact of BCG vaccine and the birth rate of the population in the dynamics of spread of TB. Assuming that all parameters are the same in both patches except the vaccine rate, we observe an increase in the number of infected if the vaccination rate decreases (Figure 2). Also taking the same parameters except birth rates, we see an increased number of infected if the rate increases (Figure 3).

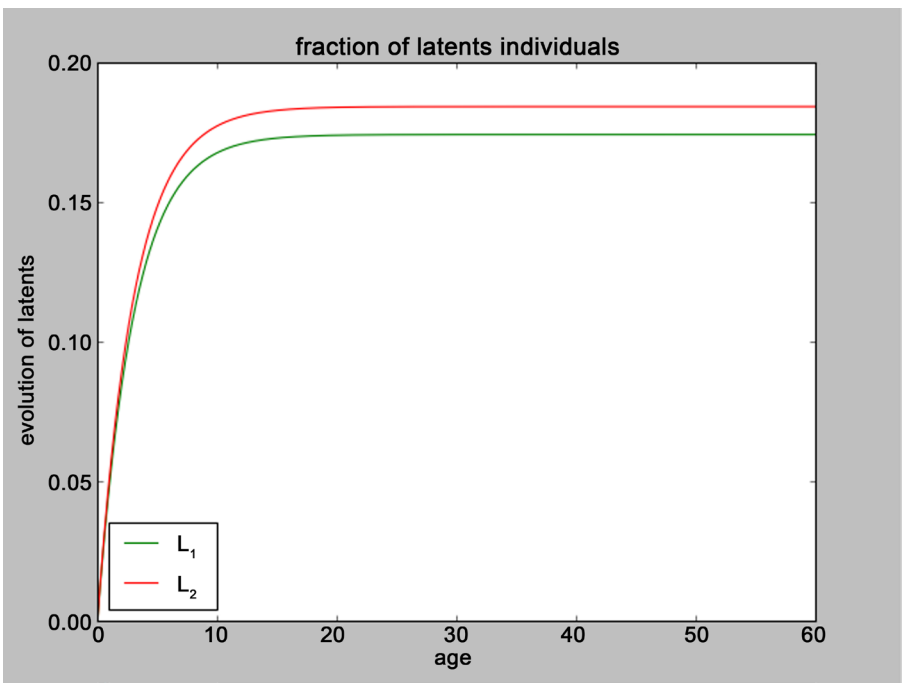

Figure 2. Evolution of the number of latents individuals with $\psi_1(a)=0.462$ and $\psi_2(a)=0.365$.

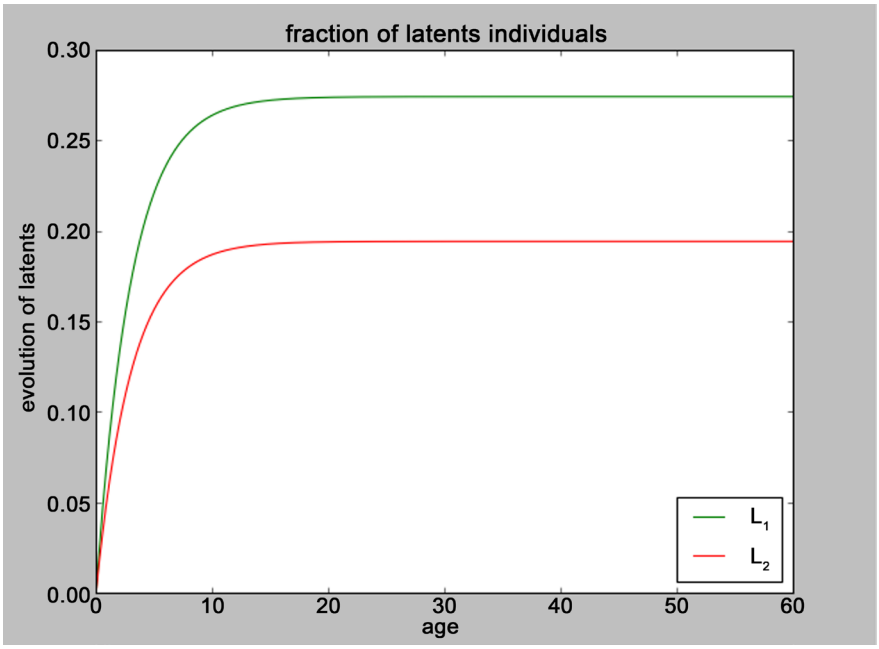

Figure 3. Evolution of the number of latents individuals with $b_1(a) = 0.0461$ and $b_2(a) = 0.0318$.

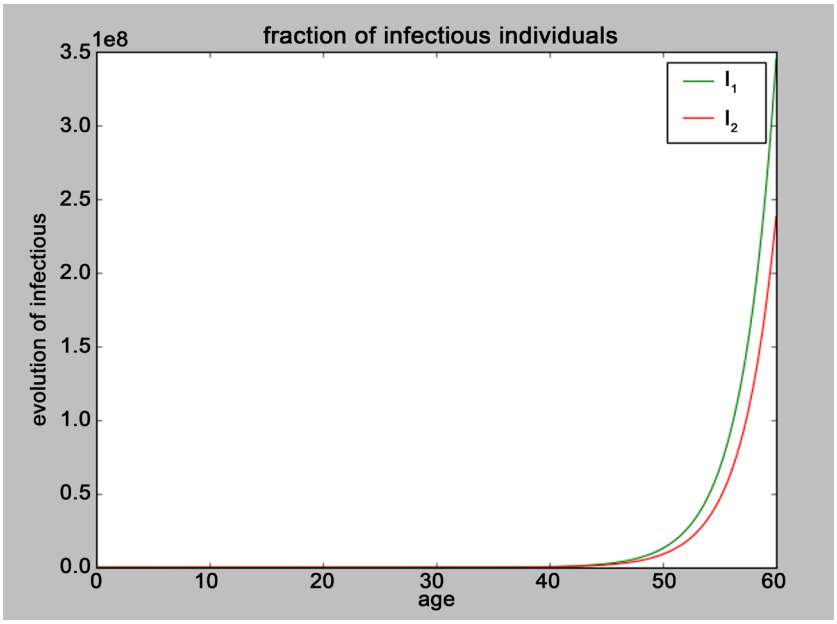

Figure 4. Evolution of the number of infectious individuals when: $\Re^1(\psi_1) = 0.572$ and $\Re^2(\psi_2) = 0.435$ ($\psi_2 = 0.852$ and $\psi_1 = 0.785$).

When $\Re^1(\psi_1) = 0.572$ and $\Re^2(\psi_2) = 0.435$ ($\psi_2 = 0.852$ and $\psi_1 = 0.785$), we have the evolution of the number of infectious individuals (Figure 4).

6. Discussion, Conclusion and Future Work

In this paper, an age structured model of two-patch for tuberculosis was analyzed and

discussed. Each sub-population is subjected to a vaccination program. Apart from age; the vaccinated compartment, we introduced as a class of treated in the model proposed by Tewa J. Jules in [11] and allowed the migration of vaccinated population. The same result was found if the most susceptible migrated too. Although some studies have shown an ineffectiveness of BCG in the prevention of tuberculosis [21], our work demonstrated the contribution of BCG in the process of eradicating TB. The negative impact of the increase in the birth rate was shown. If we neglect the mortality death rate linked to the disease, we obtain the only usual condition of global stability to the disease free equilibrium *i.e.* $\Re_0 < 1$. It remains for us many challenges such as the endemic equilibrium points of this model and the one of [8] to deal with. For future work, in order to study the real impact of the tuberculosis migration in the dynamic of the expansion of the disease, we will use this model and authorize the migration of all individuals (*i.e.* susceptible, infected, infectious, vaccinated and treated).

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