

Explicit Solutions of the Coupled mKdV Equation by the Dressing Method via Local Riemann-Hilbert Problem

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Abstract

We study the coupled mKdV equation by the dressing method via local Riemann-Hilbert problem. With the help of the Lax pairs, we obtain the matrix Riemann-Hilbert problem with zeros. The explicit solutions for the coupled mKdV equation are derived with the aid of the regularization of the Riemann-Hilbert problem.

Keywords

Coupled mKdV Equations, Riemann-Hilbert Problem, the Dressing Method

1. Introduction

The coupled mKdV equation

$$\begin{aligned} u_t - u_{xxx} + 6uvu_x &= 0 \\ v_t - v_{xxx} + 6uvv_x &= 0 \end{aligned} \quad (1)$$

is an important member of the AKNS hierarchy [1]. Moreover, it has various applications in mathematical and physical fields. In [2], Prof. Geng has given its quasi-periodic solution by using algebra-geometric methods. The equation can be solved by the method of the inverse scattering transformation, Hirota direct method, Lax pairs nonlinearization approach and others [3]-[6]. There are a lot of references for the topic [7]-[14].

In this paper, we study the Equation (1) with the help of the Riemann-Hilbert method following [15] [16]. The present paper is organized as follows. In section 2, we give the Jost solution of the spectral equation. In section 3, we discuss the analytic property of the Jost solution. In section 4, we give the Matrix Riemann-Hilbert Problem. In section 5, we obtain the soliton-solution of the coupled KdV Equation (2), and we drop

the curve of the solutions with the aid of the Matlab.

2. Jost Solution

First, we consider the coupled KdV equation

$$\begin{aligned} u_t &= u_{xxx} - 6uvu_x, \\ v_t &= v_{xxx} - 6uvv_x. \end{aligned} \quad (2)$$

As is well known [2], the Equation (2) can be derived as the compatibility of the system

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad (3)$$

where the 2×2 matrices U and V of the form

$$U = -\frac{1}{2}ik\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \quad (4)$$

$$V = \frac{1}{2}ik^3\sigma_3 - k^2Q - ikQ^2\sigma_3 + Q_{xx} + QQ_x - Q_xQ - 2Q^3 \quad (5)$$

where k is an arbitrary constant spectral parameter.

When $u = 0, v = 0$, we obtain the special solution of Equation (3). For convenience, we denote the special solution as $F = \exp\left(-\frac{1}{2}ikx\sigma_3\right)$. Then, the spectral Equation (3) is transformed into

$$H_x = -\frac{1}{2}ik[\sigma_3, H] + QH, \quad (6)$$

where, $H = \Psi F^{-1}$.

In what follows, we study the Jost solutions $H_{\pm}(x, k)$ of the Equation (6) satisfying the asymptotic conditions $H_{\pm} \rightarrow I$, at $x \rightarrow \pm\infty$. Since $\text{tr}U = 0$, these boundary conditions guarantee that $\det H_{\pm} = 1$ for all x .

In fact, the Jost functions H_{\pm} are not mutually independent. They are interconnected by the scattering matrix $S(k)$:

$$H_- = H_+ F S F^{-1}, \quad S(k) = \begin{pmatrix} a(k) & -\bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \quad \det S(k) = 1. \quad (7)$$

3. Analysis Solutions

Let us rewrite the spectral Equation (6) with the boundary conditions in the integral form:

$$\begin{aligned} (H_-)_{11}(x, k) &= I + \int_{-\infty}^x u(\xi)(H_-)_{21}(\xi, k) d\xi, \\ (H_-)_{21}(x, k) &= \int_{-\infty}^x v(\xi)(H_-)_{11}(\xi, k) \exp[ik(x - \xi)] d\xi, \end{aligned} \quad (8)$$

for the first column entries of the Jost matrix H_- .

It is easy to know that the exponent in (8) decreases for $\text{Im}k > 0$. The first column $H_-^{[1]}$ of the matrix H_- is analytic in the upper half plane and continuous on the real axis $\text{Im}k = 0$. Similarly, we know that the second column $H_+^{[2]}$ of the matrix H_+ is

analytic as well in the same domain. Then, we give a solution of Equation (6):

$$\Omega_+(x, k) = \begin{pmatrix} H_-^{[1]} & H_+^{[2]} \end{pmatrix}.$$

It can be seen that it is analytic as a whole in the upper half plane.

The analytic solution $\Omega_+(x, k)$ can be expressed in terms of the Jost function. In view of (7), we derive

$$\Omega_+ = \begin{pmatrix} H_-^{[1]} & H_+^{[2]} \end{pmatrix} = \begin{pmatrix} aH_+^{[1]} + b(k)e^{ikx} & H_+^{[2]} \end{pmatrix} = \begin{pmatrix} H_+^{[1]} & H_+^{[2]} \end{pmatrix} \begin{pmatrix} a(k) & 0 \\ b(k)e^{ikx} & 1 \end{pmatrix} \quad (9)$$

with

$$S_+ = \begin{pmatrix} a(k) & 0 \\ b(k) & 1 \end{pmatrix}. \quad (10)$$

In the same way,

$$\Omega_+ = H_- F S_- F^{-1}, \quad S_- = \begin{pmatrix} 1 & \bar{b}(k) \\ 0 & a(k) \end{pmatrix}, \quad S_+ = S S_-. \quad (11)$$

It follows from the above formalism that

$$\det \Omega_+(x, k) = a(k). \quad (12)$$

In what follows, we define a function $\Omega_-^{-1}(x, k) = \Omega_+^+(x, \tilde{k})$. It is obvious that

$$\Omega_-^{-1} = \begin{pmatrix} (H_-)_{[1]}^{-1} \\ (H_+)_{[2]}^{-1} \end{pmatrix}.$$

Then, $\Omega_-^{-1}(x, k)$ is a solution of the adjoint spectral problem. On the real axis

$$\Omega_-^{-1}(x, k) = \Omega_+^+(x, \tilde{k}) = F S_+^+ F^{-1} H_+^{-1} = F S_-^+ F^{-1} H_-^{-1},$$

and $\det \Omega_-^{-1}(x, k) = \bar{a}(\tilde{k})$, $\Omega_+(x, k)$ has an asymptotic expansion as follows:

$$\Omega_+(x, k) = I + \frac{1}{k} \Omega_+^{(1)}(x) + O\left(\frac{1}{k^2}\right), \quad (13)$$

and substitute it into the spectral Equation (6). Comparing with powers of k , we derive

$$\mathcal{Q} = i \left[\sigma_3, \Omega_+^{(1)} \right]. \quad (14)$$

In order to solve the coupled KdV Equation (2), we should find the analytic solution Ω_+ .

4. Matrix RH Problem

Through tedious calculation, we obtain RH problem

$$\Omega_-^{-1}(x, k) \Omega_+(x, k) = F \Lambda_0(k) F^{-1}, \quad \Lambda_0 = S_+^\dagger S_+ = \begin{pmatrix} 1 & \bar{b} \\ a & 1 \end{pmatrix}, \quad (15)$$

with $|a|^2 + |\bar{b}|^2 = 1$, $\text{Im} k = 0$.

It is easy to know that $\Omega_-^{-1}(x, k) \Omega_+(x, k)$ only depends on k , the x -dependence

being given by the simple exponential function F . Moreover, it is obvious that $\Omega_+(x, k) \rightarrow F$ for $k \rightarrow \infty$, in view of (12).

In order to obtain the soliton solution of the coupled KdV equation, we suppose that the zeros of $a(k)$ and $\bar{a}(k)$ are simple and finite number. We know that determinants of the matrices Ω_+ and Ω_-^{-1} are given by $a(k)$ and $\bar{a}(k)$. We assume that

$$\det \Omega_+(k_j) = 0, \quad \text{Im} k_j > 0, j = 1, 2, \dots, N,$$

$$\det \Omega_-^{-1}(\tilde{k}_j) = 0, \quad \text{Im} \tilde{k}_j < 0, l = 1, 2, \dots, N.$$

In this case, the RH problem (15) with zeros can be solved in view of its regulation.

To obtain the relevant regular problem, let us introduce a rational matrix function

$$\chi_j^{-1} = I + \frac{k_j - \tilde{k}_j}{k - k_j} \Upsilon_j, \quad \Upsilon_j = \frac{|Y_j\rangle\langle Y_j|}{\langle Y_j|Y_j\rangle},$$

where the eigenvector $|Y_j\rangle$ solves $\Omega_+(k_j)|Y_j\rangle = 0$.

Here Υ_j is the rank 1 projector $\Upsilon_j^2 = \Upsilon_j$, and $\langle Y_j| = |Y_j\rangle^+$.

In view of (11), we know that $\det \Omega_+(k)(k - k_j) = 0$ near the point k_j . We obtain $\det(\Omega_+\chi_j^{-1}) \neq 0$ at the point k_j . The matrix function Ω_-^{-1} will be regularized by the rational function

$$\chi_l = I - \frac{k_l - \tilde{k}_l}{k - \tilde{k}_l} \Upsilon_l,$$

it is easy to know that the matrix $\chi_l\Omega_-^{-1}$ has no zeros in \tilde{k}_l .

The regularization of all the other zeros is performed similarly, and eventually we obtain the following representation for the analytic solutions:

$$\Omega_{\pm} = \omega_{\pm} \Gamma, \quad \Gamma = \chi_N \chi_{N-1} \cdots \chi_1, \quad (16)$$

where the rational matrix function $\Gamma(x, k)$ accumulates all zeros of the RH problem, while the matrix functions ω_{\pm} solve the regular RH problem (without zeros)

$$\omega_-^{-1}(x, k) \omega_+(x, k) = \Gamma(x, k) F \Lambda_0(k) F^{-1} \Gamma^{-1}(x, k), \quad (17)$$

with $\Lambda_0(k) = I$, thus $\Omega_+ = \Gamma(x, k)$.

The matrix Γ will be called the dressing factor. It follows from (16) that the asymptotic expansion for the dressing factor is written as

$$\Gamma(x, k) = I + \frac{1}{k} \Gamma^{(1)}(x) + O\left(\frac{1}{k^2}\right). \quad (18)$$

We note that the dress matrix $\Gamma(k)$ can be written as

$$\begin{aligned} \Gamma(k) &= \left(I - \frac{k_N - \tilde{k}_N}{k - \tilde{k}_N} \Upsilon_N \right) \cdots \left(I - \frac{k_1 - \tilde{k}_1}{k - \tilde{k}_1} \Upsilon_1 \right) \\ &= I - \sum_{l=1}^N \frac{k_l - \tilde{k}_l}{k - \tilde{k}_l} |x_l\rangle\langle y_l|, \\ \Gamma^{-1}(k) &= \left(I + \frac{k_N - \tilde{k}_N}{k - \tilde{k}_N} \Upsilon_N \right) \cdots \left(I + \frac{k_1 - \tilde{k}_1}{k - \tilde{k}_1} \Upsilon_1 \right) \end{aligned} \quad (19)$$

$$= I + \sum_{l=1}^N \frac{k_l - \tilde{k}_l}{k - \tilde{k}_l} |y_l\rangle \langle x_l|. \quad (20)$$

Thus, we derived $2N$ vectors $|x_j\rangle$ and $|y_j\rangle$ instead of N vectors $|Y_j\rangle$. It is obvious that $\Gamma(k)\Gamma^{-1}(k) = I$ at the point $k = k_j$. To avoid divergence at $k \rightarrow k_j$, we should pose $\Gamma(k_j)|y_j\rangle\langle x_j| = 0$, that is

$$\left(I - \frac{k_l - \tilde{k}_l}{k_j - \tilde{k}_l} |y_l\rangle \langle x_l| \right) |y_j\rangle \langle x_j| = 0. \quad (21)$$

We note that the matrix $\Gamma(k)$ can be decomposed into the following form:

$$\Gamma(k) = I - \sum_{j,l=1}^N \frac{1}{k - \tilde{k}_l} |j\rangle \langle l| (\mathcal{D}^{-1})_{jl}, \quad \mathcal{D}_{lj} = \frac{\langle l|j\rangle}{k_j - \tilde{k}_l}, \quad (22)$$

where $\mathcal{D} = \mathcal{D}_{lj} = \langle y_l | \frac{1}{k_j - \tilde{k}_l} | y_j \rangle$. Similarly,

$$\Gamma(k) = I - \sum_{j,l=1}^N \frac{1}{k - k_j} |j\rangle \langle l| (\mathcal{D}^{-1})_{jl}, \quad (23)$$

where $|j\rangle \equiv |y_j\rangle$. In what follows, we rewrite (13) as

$$Q = i[\sigma_3, \Gamma^{(1)}(x)]. \quad (24)$$

Let us differentiate the equation $\Omega_+(k_j)|j\rangle = 0$ in x , and in view of (6), we derive

$$\partial_x \Omega_+(x, k_j) |j\rangle + \Omega_+(x, k_j) |j\rangle_x = i \frac{k_j}{2} \Omega_+(x, k_j) |j\rangle + \Omega_+(x, k_j) |j\rangle_x = 0,$$

thus, we have

$$|j\rangle_x = -\frac{1}{2} i k_j \sigma_3 |j\rangle. \quad (25)$$

In the same way, we obtain the evolutionary equation

$$|j\rangle_t = \frac{1}{2} i k_j^2 \sigma_3 |j\rangle. \quad (26)$$

In this end, we establish explicitly the vector $|j\rangle$ as

$$|j\rangle = \exp \left[\left(-i \frac{1}{2} k_j x + i \frac{1}{2} k_j^2 t \right) \sigma_3 \right] |j_0\rangle, \quad (27)$$

where $|j_0\rangle$ is a vector integration constant.

Similarly, according to $\langle \tilde{j} | \Omega_-^{-1}(\tilde{k}_j) = 0$, we obtain the solution

$$\langle \tilde{j} | = \exp \left[\left(-i \frac{1}{2} \tilde{k}_j x + i \frac{1}{2} \tilde{k}_j^2 t \right) \sigma_3 \right] \langle \tilde{j}_0 |, \quad (28)$$

where $\langle \tilde{j}_0 |$ is a vector integration constant.

5. One Soliton Solution

We consider the case $N = 1$ and pose $k_1 = \xi + i\eta$, $\tilde{k}_1 = \tilde{\xi} + i\tilde{\eta}$. Then, we have

$$|1\rangle = \begin{pmatrix} \exp \frac{\eta x}{2} x + \frac{1}{2(\xi^2 - \eta^2)t} + i \left[-\frac{\xi}{2} x + \xi \eta t \right] p_1 \\ \exp -\frac{\eta x}{2} x - \frac{1}{2(\xi^2 - \eta^2)t} - i \left[-\frac{\xi}{2} x + \xi \eta t \right] p_2 \end{pmatrix}, \quad (29)$$

where, p_1, p_2 are components of the constant vector $|1_0\rangle$.

$$\langle \tilde{1} | = \begin{pmatrix} \exp \frac{\tilde{\eta} x}{2} x + \frac{1}{2(\tilde{\xi}^2 - \tilde{\eta}^2)t} + i \left[-\frac{\tilde{\xi}}{2} x + \tilde{\xi} \tilde{\eta} t \right] \tilde{p}_1 \\ \exp -\frac{\tilde{\eta} x}{2} x - \frac{1}{2(\tilde{\xi}^2 - \tilde{\eta}^2)t} - i \left[-\frac{\tilde{\xi}}{2} x + \tilde{\xi} \tilde{\eta} t \right] \tilde{p}_2 \end{pmatrix}, \quad (30)$$

where, \tilde{p}_1, \tilde{p}_2 are components of the constant vector $\langle 1_0 |$.

The dress formula (19) reduced to

$$u = i\Gamma_{12}^{(1)}(x), \quad v = -i\Gamma_{21}^{(1)}(x). \quad (31)$$

At the same time, we have $\Gamma = I - \frac{k_1 - \tilde{k}_1}{k - \tilde{k}_1} \Upsilon_1$, from which, we obtain

$$\Gamma^{(1)} = -i(\eta + \tilde{\eta}) \frac{|1\rangle \langle \tilde{1}|}{\langle \tilde{1} | 1 \rangle}. \quad (32)$$

Denoting $z = \eta x + (\xi^2 - \eta^2)t + \alpha$, $\kappa = \xi(-x + 2\eta t) + \beta$, thus

$$|1\rangle = \exp \left[-\frac{\alpha + i\beta}{2} \right] \begin{pmatrix} \exp \frac{z + i\varphi}{2} \\ \exp -\frac{z + i\varphi}{2} \end{pmatrix}. \quad (33)$$

In the same way, defining $\tilde{z} = \tilde{\eta} x + (\tilde{\xi}^2 - \tilde{\eta}^2)t + \tilde{\alpha}$, $\tilde{\varphi} = \tilde{\xi}(-x + 2\tilde{\eta}t) + \tilde{\beta}$, thus

$$\langle \tilde{1} | = \exp \left[-\frac{\tilde{\alpha} + i\tilde{\beta}}{2} \right] \begin{pmatrix} \exp \frac{\tilde{z} + i\tilde{\varphi}}{2} \\ \exp -\frac{\tilde{z} + i\tilde{\varphi}}{2} \end{pmatrix}. \quad (34)$$

Substituting (31) and (32) into (30), we have

$$|1\rangle \langle \tilde{1}| = \exp \left(-\frac{\alpha + \tilde{\alpha}}{2} \right) \begin{pmatrix} \exp \frac{z + \tilde{z}}{2} & \exp \frac{i\varphi + i\tilde{\varphi}}{2} \\ \exp -\frac{i\varphi + i\tilde{\varphi}}{2} & \exp -\frac{z + \tilde{z}}{2} \end{pmatrix}.$$

Moreover, $\langle 1 | 1 \rangle = \exp \left(-\frac{\alpha + \tilde{\alpha}}{2} \right) (\exp z + \exp -\tilde{z})$, hence,

$$\Gamma^{(1)} = -i(\eta + \tilde{\eta}) \begin{pmatrix} \exp \frac{z + \tilde{z}}{2} & \exp i(\varphi + \tilde{\varphi}) \\ \exp -i(\varphi + \tilde{\varphi}) & \exp -\frac{z + \tilde{z}}{2} \end{pmatrix} \sec(z + \tilde{z}).$$

From which, we have the solutions of the coupled KdV Equation (2)

$$u = (\tilde{\eta} + \eta)e^{i\tilde{\varphi} + \varphi} \sec(z + \tilde{z}), \quad v = -(\tilde{\eta} + \eta)e^{-i\tilde{\varphi} + \varphi} \sec(z + \tilde{z}). \quad (35)$$

Here, ξ , $\tilde{\xi}$, η and $\tilde{\eta}$ determine the soliton velocity and amplitude, respectively, while α , β , $\tilde{\alpha}$ and $\tilde{\beta}$ give the initial position and phase of the soliton. In what follows, we plot the graph for $u(x, t)$ in order to analyze the solutions (35). **Figure 1** and **Figure 2** are the imaginary part and real part of $u(x, t)$, respectively. From the two solution curves, we can see that the difference between the real and imaginary part.

In the same way, we drop the solution curves of v for **Figure 3** and **Figure 4**.

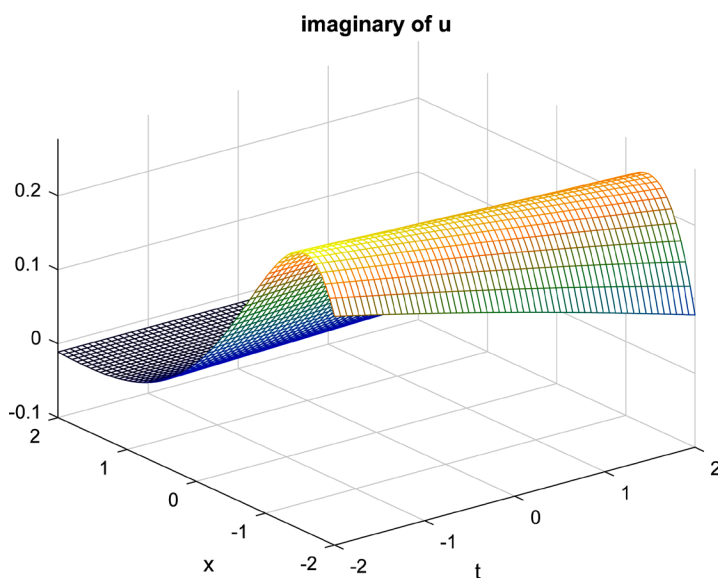


Figure 1. The soliton solution curve of imaginary part of $u(x, t)$ for $\eta = 0.01$, $\tilde{\eta} = 0.5$, $\xi = 0.05$, $\tilde{\xi} = 0.07$, $\alpha = 0.2$, $\tilde{\alpha} = 0.4$, $\beta = 0.6$, $\tilde{\beta} = 0.9$, $x \in [-2, 2]$, $t \in [-6, 6]$.

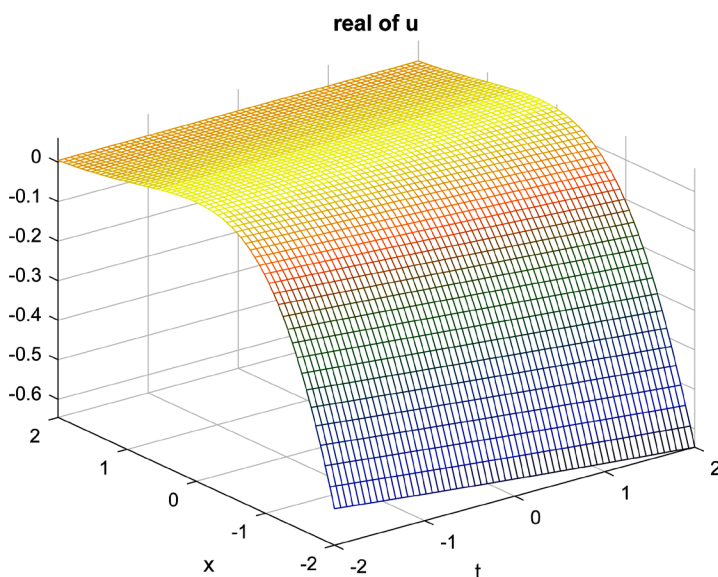


Figure 2. The soliton solution curve of real part of $u(x, t)$ for $\eta = 0.01$, $\tilde{\eta} = 0.5$, $\xi = 0.05$, $\tilde{\xi} = 0.07$, $\alpha = 0.2$, $\tilde{\alpha} = 0.4$, $\beta = 0.6$, $\tilde{\beta} = 0.9$, $x \in [-2, 2]$, $t \in [-6, 6]$.

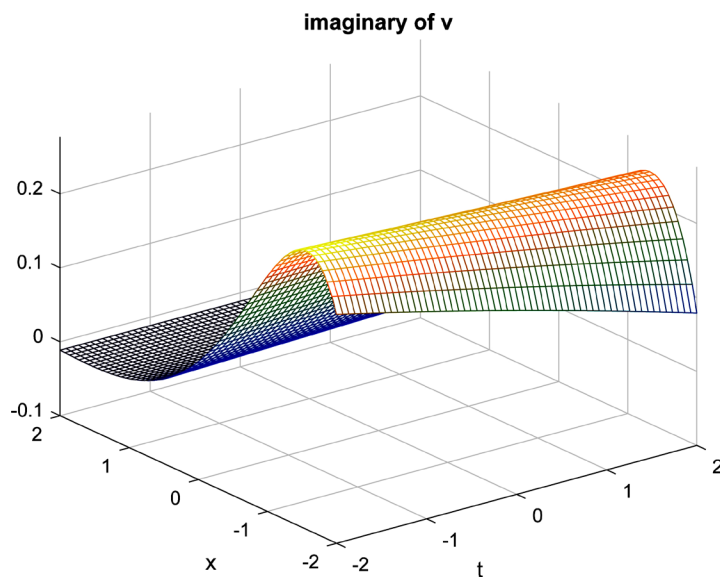


Figure 3. The soliton solution curve of imaginary part of $v(x,t)$ for $\eta=0.01$, $\tilde{\eta}=0.5$, $\xi=0.05$, $\tilde{\xi}=0.07$, $\alpha=0.2$, $\tilde{\alpha}=0.4$, $\beta=0.6$, $\tilde{\beta}=0.9$, $x \in [-2,2]$, $t \in [-6,6]$.

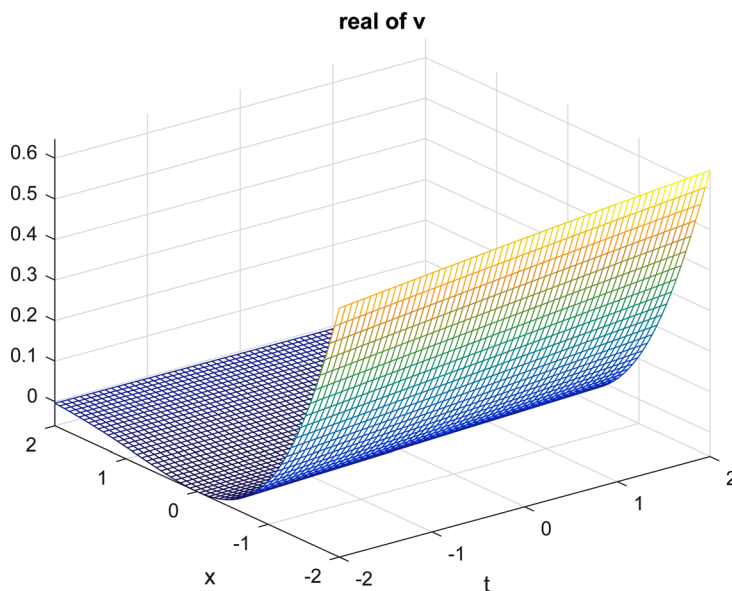


Figure 4. The soliton solution curve of real part of $v(x,t)$ for $\eta=0.01$, $\tilde{\eta}=0.5$, $\xi=0.05$, $\tilde{\xi}=0.07$, $\alpha=0.2$, $\tilde{\alpha}=0.4$, $\beta=0.6$, $\tilde{\beta}=0.9$, $x \in [-2,2]$, $t \in [-6,6]$.

From the graphs, it is shown that u and v have the similar solution form. The difference exists between the real and imaginary part. In fact, we chose different parameters, and the solution curves between the real part and imaginary part had corresponding changes.

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