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Stability Analysis of SIQS Epidemic Model with Saturated Incidence Rate

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Abstract

A SIQS epidemic model with saturated incidence rate is studied. Two equilibrium points exist for the system, disease-free and endemic equilibrium. The stability of the disease-free equilibrium and endemic equilibrium exists when the basic reproduction number R_0, is less or greater than unity respectively. The global stability of the disease-free and endemic equilibrium is proved using Lyapunov functions and Poincare-Bendixson theorem plus Dulac's criterion respectively.

Keywords

SIQS Epidemic Model, Saturated Incidence Rate, Basic Reproduction Number, Lyapunov Function, Poincare-Bendixson, Dulac Criterion

1. Introduction

The isolation and treatment of symptomatic individuals coupled with the quarantining of individuals that have a high risk of having been infected, constitute two commonly used epidemic control measures. Mass quarantine can inflict significant social, psychological and economic costs without resulting in the detection of many infected individuals. Day *et al*. [1], Hethcote *et al*. [2] considered SIQS and SIQR epidemic models with three forms of incidence, which include the bilinear, standard and quarantined-adjusted incidences.

Feng and Thieme [3] considered SEIQR models with arbitrarily distributed periods of infection, including quarantine and a general incidence assumed that all infected individuals go through the quarantine stage and investigated the model dynamics. Settapat and Wirawah [4] discussed the SIQ epidemic model with constant immigration. Yang *et al*. [5] also studied an SIQ epidemic model with isolation and nonlinear incidence rate.

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El-Marouf and Alihaby [6] studied the equilibrium points and their local stability for SIQ and SIQR epidemic models with three forms of incidence rates. They also studied the global stability of the equilibrium by constructing the new forms of Lyapunov functions.

Gbadamosi and Adebimpe investigated an SIQ epidemic model with nonlinear incidence rate. They introduced the concept that describes the present and past states of the disease.

We extended the work of Gbadamosi and Adebimpe to include the rates at which individuals recover and return to susceptible compartment from compartments I and Q respectively and we apply Lyapunov functions and Poincare-Bendixson theorem plus Dulac's criterion to prove the global stability of disease-free and endemic equilibria respectively.

2. The Model

The model that governs a system of differential equation is presented as follows:

$$\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= (1-p)A - \frac{\beta SI}{1+mI} - dS + \gamma I + \varepsilon Q \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta SI}{1+mI} + pA - (\gamma + \delta + d + \alpha) I \\ \frac{\mathrm{d}Q}{\mathrm{d}t} &= \delta I - (\varepsilon + d + \alpha) Q \end{aligned} \tag{1}$$

Subject to initial conditions

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad Q(0) = Q_0 \geq 0 \tag{2}$$

The parameters with their descriptions are presented in **Table 1**.

The addition of the system (1), gives

$$\frac{\mathrm{d}N}{\mathrm{d}t} = A - dN - \alpha I - \alpha Q \quad \text{where } N = S + I + Q$$

From above equation, we get

$$\begin{aligned} &0 \leq \limsup_{t\to\infty} N(t) \leq N_0 \\ &\text{with } \limsup_{t\to\infty} N(t) = N_0 \ \text{ if and only if } \limsup_{t\to\infty} I(t) = 0 \end{aligned}$$

From the first equation of the system (1), it follows

$$0 \leq \limsup_{t\to\infty} S(t) \leq S_0$$

And the second equation gives

Table 1. Descriptions of parameters.

Parameter	Description
$(1-p)A$	Recruitment of the susceptible corresponding to Births and Immigration
pA	constant recruitment of the infected compartment
D	Per capita natural mortality rate
δ	The rate constant for individuals leaving the compartment I for the quarantine compartment Q
α	Disease-related death rate constant in compartments I and Q
γ	Rate at which individuals recover and return to Susceptible (S) from compartment I
ε	Rate at which individuals recover and return to Susceptible (S) from compartment Q
M	The saturation constant

$$0 \le \limsup_{t\to\infty} I(t) \le I_0 .$$

So, from the above, if $N > N_0$, then $\frac{\mathrm{d}N}{\mathrm{d}t} < 0$.

We can now write

$$\Omega = \left\{ (S, I, Q) \in R_+^3 : S + I + Q \le N_0, S \le S_0, I \le I_0 \right\}$$

Equilibria

The system (1) has always the disease-free equilibrium at $E_0 = (S_0, I_0, Q_0) = \left(\frac{(1-p)A}{d}, 0, 0 \right)$

Endemic Equilibrium: $E_* = (S_*, I_*, Q_*)$

3. Local Stability

In this section, we discussed the local stability of the disease-free equilibrium and endemic equilibrium for the system (1).

We state and prove the following results:

Theorem 1: At E_0, the disease-free equilibrium of the system (1) is locally asymptotically stable when $R_0 < 1$.

Proof: The Jacobian matrix at the point E_0 through linearization is given by

$$J_0 = \begin{pmatrix} -d & -(\beta S_0 - \gamma) & \varepsilon \\ 0 & \beta S_0 - (\gamma + \delta + d + \alpha) & 0 \\ 0 & \delta & -(\varepsilon + d + \alpha) \end{pmatrix}$$

By finding the eigenvalues, we have the following λs:

$$\lambda_1 = -d, \quad \lambda_2 = \beta S_0 - (\gamma + \delta + d + \alpha), \quad \lambda_3 = -(\varepsilon + d + \alpha)$$

For λ_2 to be negative $\beta S_0 < (\gamma + \delta + d + \alpha)$

That is, $\frac{\beta S_0}{\gamma + \delta + d + \alpha} < 1$

Let $R_0 = \frac{\beta (1-p) A}{d(\gamma + \delta + d + \alpha)}$

If $R_0 = \frac{\beta (1-p) A}{d(\gamma + \delta + d + \alpha)} < 1, \ \lambda_2 < 0$

Since $\lambda_1 < 0, \lambda_3 < 0$ and $\lambda_2 < 0$ if $R_0 < 1$, the disease-free equilibrium is locally asymptotically stable.

Theorem 3.1: The system (1) is locally asymptotically stable at E_* if $R_0 > 1$, otherwise unstable.

Proof: At the endemic equilibrium E_*, the Jacobian matrix of the system (1) is given by:

$$J_* = \begin{pmatrix} -(\beta I_* + d) & -\beta S_0 + \gamma & \varepsilon \\ \beta I_* & \beta S_* - (\gamma + \delta + d + \alpha) & 0 \\ 0 & \delta & -(\varepsilon + d + \alpha) \end{pmatrix}$$

The characteristic equation of the Jacobian matrix J_* is given by

$$\lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3$$

where $a_1 = 3\alpha + 2\delta + \gamma + 3d + \beta I_* + \varepsilon - 2\beta S_*$

$$\begin{aligned} a_2 = {} & 2\beta I_* \alpha + \beta I_* d + \beta I_* \delta + 3d^2 + 4\alpha d + \alpha^2 + \gamma\alpha + \delta\alpha + 2\delta d + 2\gamma d \\ & + 3d\varepsilon + \gamma\delta + \beta I_* \varepsilon + \beta I_* d - \beta S_* d - 2\beta S_* \varepsilon \end{aligned}$$

$$a_3 = \beta I_* d\varepsilon + \beta I_* \alpha\varepsilon + \gamma d\varepsilon + \delta d\varepsilon + d^2\varepsilon + \alpha d\varepsilon + \beta I_* \delta d + \beta I_* \alpha d + \gamma d^2 + d^3 + 2\alpha d^2$$
$$+ \beta I_* \alpha\delta + \beta I_* d\alpha + \beta I_* \alpha^2 + \gamma\alpha d + \delta\alpha d + \alpha^2 d - \beta S_* \alpha d$$

as $a_1 > 0, a_2 > 0, a_3 > 0$ and $a_1 a_2 - a_3 > 0$

If $a_1 a_2 > a_3$, by Routh Hurwitz criterion, all the eigenvalues of the system (1) has negative real part. Therefore, the endemic equilibrium of the system (1) at E_* is locally asymptotically stable.

4. Global Stability

In this section, we study the global stability of the disease-free equilibrium and endemic equilibrium by Lyapunov function and Poincare-Bendixson theorem respectively.

Theorem 3: (Dulac's Criterion)

Consider the following general nonlinear autonomous system of de

$$x(t) = f(x), x \in E \tag{*}$$

Let $f = C^1(E)$ where E is a simple connected region in R^2. If the exists a function it $H \in C^1(E)$ such that $\nabla \cdot (Hf)$ is not identically zero and does not change sign in E, the system (*) has no close orbit lying entirely in E. if A is an annular region contained in E on which $\nabla \cdot (Hf)$ does not change sign, then there is at most one limit cycle of the system (*) in A.

Theorem 4: (The Poincare-Bendixson Theorem): Suppose that $f \in C^1(E)$ where E is an open subset of R^n and that the system (*) has a rejecting Γ contained in a compact subset f of E. assume that the system (*) has only one unique equilibrium point x_0 in f, then one of the following possibilities holds.

(a) $w(\Gamma)$ is the equilibrium point x_0

(b) $w(\Gamma)$ is a periodic orbit

(c) $w(\Gamma)$ is a graphic

Theorem 5: The disease-free equilibrium of the model (1) is globally asymptotically stable if $R_0 < 1$

Proof: To prove this result, we construct the following Lyapunov function

$$L = u_1(S - S_0) + u_2(I - I_0) + u_3 Q \tag{3}$$

where u_1, u_2 and u_3 are positive constants to be determined later. Differentiating equation (3) with respect to t, we obtain

$$L' = u_1\left[(1-p)A - \frac{\beta SI}{1+mI} - dS + \gamma I + \varepsilon Q\right] + u_2\left[\frac{\beta SI}{1+mI} + pA - (\gamma + \delta + d + \alpha)I\right]$$
$$+ u_3\left[\delta I - (\varepsilon + d + \alpha)Q\right]$$

After rearrangements, we get

$$L' = \frac{\beta SI}{1+mI}(u_2 - u_1) + pA(u_2 - u_1) + \gamma I(u_2 - u_1) + \varepsilon Q(u_3 - u_1) + \delta I(u_3 - u_1)$$
$$+ u_1 A - u_1 dS - u_2 dI - u_3 dQ - u_2 \alpha I - u_3 \alpha Q$$

Let us choose the constants $u_1 = u_2 = u_3 = 1$. Finally, we obtain

$$L' = A - d(S + I + Q) - \alpha(I + Q)$$
$$L' = -(dN - A) - \alpha(N - S) < 0$$

Thus, the disease-free equilibrium of the system (1) is globally asymptotically stable if $R_0 < 1$

In the next theorem, we present the global stability of the endemic equilibrium of the system (1) at E_*

Theorem 6: The endemic equilibrium E_* of the system (1) is globally asymptotically stable if $R_0 > 1$.

Proof: In order to prove the result, we use Dulac plus Poincare Bendixson theorem as follow

$$H(S, I, Q) = \frac{1}{SIQ} \text{ where } S > 0,\ I > 0, Q > 0.$$

Then,

$$\nabla\cdot(HF)=\frac{\partial}{\partial S}(H\cdot F_1)+\frac{\partial}{\partial I}(H\cdot F_2)+\frac{\partial}{\partial Q}(H\cdot F_3)$$

$$=\frac{\partial}{\partial S}\left[\frac{1}{SIQ}\left((1-p)A-\frac{\beta SI}{1+mI}-dS+\gamma I+\varepsilon Q\right)\right]+\frac{\partial}{\partial I}\left[\frac{\beta SI}{1+mI}+pA-(\gamma+\delta+d+\alpha)\ I\right]$$

$$+\frac{\partial}{\partial Q}\left[\delta I-(\varepsilon+d+\alpha)Q\right]$$

$$\nabla\cdot(HF)=-\frac{(1-p)A}{S^2IQ}-\frac{\gamma}{S^2Q}-\frac{\varepsilon}{S^2I}-\frac{pA}{SI^2Q}-\frac{\delta}{SQ^2}<0$$

Hence, by Dulac's criterion, there is no closed orbit in the first quadrant. Therefore, the endemic equilibrium is globally asymptotically stable.

5. Discussion of Results

The mathematical and stability analysis of SIQS epidemic model with saturated incidence rate and temporary immunity has been presented. We investigated the local stability of the disease-free equilibrium and endemic equilibrium using the basic reproduction number, R_0. We observed that, when $R_0<1$, the disease-free equilibrium is stable at E_0 locally and endemic equilibrium is unstable which means there is tendency for the disease to die out in the long run. We proved the global stability of the disease free equilibrium and endemic equilibrium of the model using Lyapunov function and Dulac's criterion plus Poincare-Bendixson theorem respectively.

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