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Global Attractor for a Class of Nonlinear Generalized Kirchhoff-Boussinesq Model

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Abstract

In this paper, we study the long time behavior of solution to the initial boundary value problem for a class of Kirchhoff-Boussinesq model flow $u_{tt}+\alpha u_t-\beta\Delta u_t+\Delta^2 u=div\left(g\left(|\nabla u|^2\right)\nabla u\right)+\Delta h(u)+f(x)$. We first prove the wellness of the solutions. Then we establish the existence of global attractor.

Keywords

Kirchhoff-Boussinesq Model, Strongly Damped, Existence, Global Attractor

1. Introduction

In this paper, we are concerned with the existence of global attractor for the following nonlinear plate equation referred to as Kirchhoff-Boussinesq model:

$$u_{tt}+\alpha u_t-\beta\Delta u_t+\Delta^2 u=div\left(g\left(|\nabla u|^2\right)\nabla u\right)+\Delta h(u)+f(x) \quad \text{in } \Omega\times\mathbb{R}^+ \tag{1.1}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x), \quad x\in\Omega \tag{1.2}$$

$$u(x,t)\big|_{\partial\Omega}=0,\Delta u(x,t)\big|_{\partial\Omega}=0, \quad (x)\in\Omega \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, and α,β are positive constants, and the assumptions on $g\left(|\nabla u|^2\right),h(u)$ will be specified later.

Recently, Chueshov and Lasiecka [1] studied the long time behavior of solutions to the Kirchhoff-Boussinesq plate equation

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$$u_{tt}+ku_t+\Delta^2 u=div\left[f_0\left(\nabla u\right)\right]+\Delta\left[f_1\left(u\right)\right]-f_2\left(u\right) \tag{1.4}$$

with clamped boundary condition

$$u\left(x,t\right)\big|_{\partial\Omega}=0,\quad \frac{\partial u\left(x,t\right)}{\partial v}\bigg|_{\partial\Omega}=0 \tag{1.5}$$

with $\Omega\subset\mathbb{R}^2$ where v is the unit outward normal on $\partial\Omega$. Here $k>0$ is the damping parameter, the mapping $f_0:\mathbb{R}^2\to\mathbb{R}^2$ and the smooth functions f_1 and f_2 represent (nonlinear) feedback forces acting upon the plate, in particular,

$$f_0\left(\nabla u\right)=\left|\nabla u\right|^2\nabla u,\quad f_1\left(u\right)=u^2+u.$$

When $f_0\left(\nabla u\right)=\left|\nabla u\right|^{m-1}\nabla u=\sigma\left(\left|\nabla u\right|^2\right)\nabla u$ and $f_1\left(u\right)=0$, also considering the (1.4) with a strong damping, then (1.4) becomes a class of Krichhoff models arising in elastoplastic flow,

$$u_{tt}-div\left\{\sigma\left(\left|\nabla u\right|^2\right)\nabla u\right\}-\Delta u_t+\Delta^2 u+h\left(u_t\right)+g\left(u\right)=f\left(x\right) \tag{1.6}$$

which Yang Zhijian and Jin Baoxia [2] studied. In this model, Yang Zhijian and Jin Baoxia gained that under rather mild conditions, the dynamical system associated with above-mentioned IBVP possesses in different phase spaces a global attractor associated with problem (1.6), (1.2) and (1.3) provided that g and h satisfy the nonexplosion condition,

$$\liminf_{|s|\to\infty}\frac{G\left(s\right)}{\left|s\right|^{m+1}}\geq 0 \tag{1.7}$$

$$\liminf_{|s|\to\infty}\frac{sg\left(s\right)-\rho G\left(s\right)}{\left|s\right|^{m+1}}\geq 0 \tag{1.8}$$

with $0<\rho<2$, $G\left(s\right)=\int_0^s g\left(\tau\right)\mathrm{d}\tau$, $1\leq m<\left(N/\left(N-2\right)^+\right)\left(m<\infty\right)$, and $h=h_1+h_2$ and there exist constant $\delta_1\in\left(0,1\right),\theta_1\in\left(0,\frac{1}{2}\right),\beta_1>0$ such that

$$\left(h_2\left(v\right),v\right)\geq 0,\quad \left(h_1\left(v\right),v\right)\geq-\theta_1\left[\left(h_2\left(v\right),v\right)+\left\|v\right\|_{v_1}^2\right]-\beta_1. \tag{1.9}$$

Zhijian Yang, Na Feng and Ro Fu Ma [3] also studied the global attractor for the generalized double dispersion equation arising in elastic waveguide model

$$u_{tt}-\Delta u-\Delta u_{tt}+\Delta^2 u-\Delta u_t-\Delta g\left(u\right)=f\left(x\right). \tag{1.10}$$

In this model, g satisfies the nonexplosion condition,

$$\liminf_{|s|\to\infty}\frac{g\left(s\right)}{s}\geq-\lambda_1,\quad \left|g'\left(s\right)\right|\leq C\left(1+\left|s\right|^{p-1}\right),\quad s\in\mathbb{R} \tag{1.11}$$

where $\lambda_1\left(>0\right)$ is the first eigenvalue of the $-\Delta$, and $1<p<\infty$ as $N=2$; $1\leq p\leq p^*\equiv\frac{N+2}{N-2}$ as $N\geq 3$.

T. F. Ma and M. L. Pelicer [4] studied the existence of a finite-dimensional global attractor to the following system with a weak damping.

$$u_{tt}+u_{xxxx}-\left(\sigma\left(u_x\right)\right)_x+ku_t+f\left(u\right)=h\quad \text{in } \left(0,L\right)\times\mathbb{R}^+ \tag{1.12}$$

with simply supported boundary condition

$$u\left(0,t\right)=u\left(L,t\right)=u_{xx}\left(0,t\right)=u_{xx}\left(L,t\right)=0,\quad t\geq 0 \tag{1.13}$$

and initial condition

$$u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x),\quad x\in(0,L) \tag{1.14}$$

where $\sigma(z)=|z|^{p-2}$, $p\geq 2, k>0$, and $f\in C^1(\mathbb{R})$, $-\rho\leq \hat{f}(s)=\int_0^s f(\tau)\mathrm{d}\tau\leq f(s)s,\ \rho>0,\ \forall s\in\mathbb{R}.$

For more related results we refer the reader to [5]-[8]. Many scholars assume $div\left(g\left(|\nabla u|^2\right)\nabla u\right)=\|\nabla u\|^{m-1}\nabla u$, to make these equations more normal; we try to make a different hypothesis (specified Section 2), by combining the idea of Liang Guo, Zhaoqin Yuan, Guoguang Lin [9], and in these assumptions, we get the uniqueness of solutions, then we study the global attractors of the equation.

2. Preliminaries

For brevity, we use the follow abbreviation:

$$L^p=L^p(\Omega),\quad H^k=H^k(\Omega),\quad H=L^2,\quad \|\cdot\|=\|\cdot\|_{L^2},\quad \|\cdot\|_p=\|\cdot\|_{L^p}$$

with $p\geq 1$, and $V_2=H^2\cap H_0^1$, where H^k are the L^2-based Sobolev spaces and H_0^k are the completion of $C_0^\infty(\Omega)$ in H^k for $k>0$. The notation $(\cdot,\cdot)$ for the *H*-inner product will also be used for the notation of duality pairing between dual spaces.

In this section, we present some materials needed in the proof of our results, state a global existence result, and prove our main result. For this reason, we assume that

(H_1) $g\in C^1(\Omega)$,

$$\liminf_{|s|\to\infty}\frac{G(s)}{|s|^{\frac{m+3}{2}}}\geq -C \tag{2.1}$$

$$\liminf_{|s|\to\infty}\frac{sg(s)-\rho G(s)}{|s|^{\frac{m+3}{2}}}\geq -C \tag{2.2}$$

where $G(s)=\int_0^s g(\tau)\mathrm{d}\tau$, $0<\rho<2$, and when $N\geq 2$,

$$|g'(s)|\leq C\left(1+|s|^{\frac{m-1}{2}}\right),\quad s\in\Omega, \tag{2.3}$$

where $1\leq m<\infty$ as $N=2$; $1\leq m\leq m^*\equiv\dfrac{6-N}{N-2}$ as $3\leq N\leq 4$; and $m=1$ as $N\geq 5$.

(H_2) $h\in C^1$ and $\|h'(u)\|_\infty<\dfrac{\sqrt{2}\lambda_1}{4}$, $\lambda_1(>0)$ is the first eigenvalue of the $-\Delta$.

Now, we can do priori estimates for Equation (1.1).

Lemma 1. Assume (H_1), (H_2) hold, and $(u_0,u_1)\in V_2\times H$, $f\in H$. Then the solution (u,v) of the problem (1.1)-(1.3) satisfies $(u,v)\in V_2\times H$, and

$$\|(u,v)\|_{V_2\times H}^2=\|\Delta u\|^2+\|v\|^2\leq\frac{H_1(0)}{k}\mathrm{e}^{-\alpha_1 t}+\frac{C_1}{k\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right) \tag{2.4}$$

where $v=u_t+\varepsilon u$, $0<\varepsilon<\min\left\{\dfrac{\alpha}{4},\dfrac{\lambda_1^2}{2\alpha},\dfrac{\lambda_1}{4\beta}\right\}$, and

$H_1(0)=\|v_0\|^2+\|\Delta u_0\|^2-\beta\varepsilon\|\nabla u_0\|^2+\int_\Omega\left(G\left(|\nabla u_0|^2\right)+C_\eta\right)\mathrm{d}x, v_0=u_1+\varepsilon u_0$, thus there exists E_0 and $t_1=t_1(\Omega)>0$, such that

$$\|(u,v)\|_{V_2\times H}^2=\|\Delta u\|^2+\|v\|^2\leq E_0\ (t>t_1). \tag{2.5}$$

Remark 1. (2.1) and (2.1) imply that there exist positive constants C_η and $\tilde{C}_\eta$, such that

$$G(s)\geq -C_\eta,\quad sg(s)-\rho G(s)\geq\tilde{C}_\eta. \tag{2.6}$$

Proof of Lemma 1.

Proof. Let $v = u_t + \varepsilon u$, then v satisfies

$$v_t + (\alpha - \varepsilon)v + (\varepsilon^2 - \alpha\varepsilon)u - \beta\Delta v + \beta\varepsilon\Delta u + \Delta^2 u = div\left(g\left(|\nabla u|^2\right)\nabla u\right) + \Delta h(u) + f(x). \quad (2.7)$$

Taking H-inner product by v in (2.7), we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + (\alpha - \varepsilon)\|v\|^2 + (\varepsilon^2 - \alpha\varepsilon)(u,v) + \beta\|\nabla v\|^2 + \beta\varepsilon(\Delta u, v) + (\Delta^2, v)\\ &= \left(div\left(g\left(|\nabla u|^2\right)\nabla u\right), v\right) + (\Delta h(u), v) + (f(x), v).\end{aligned} \quad (2.8)$$

Since $v = u_t + \varepsilon u$ and $0 < \varepsilon < \min\left\{\frac{\alpha}{4}, \frac{\lambda_1^2}{2\alpha}, \frac{\lambda_1}{4\beta}\right\}$, by using Holder inequality, Young's inequality and Poincare inequality, we deal with the terms in (2.8) one by one as follow,

$$(\alpha - \varepsilon)\|v\|^2 \ge \frac{3\alpha}{4}\|v\|^2 \quad (2.9)$$

$$\begin{aligned}(\varepsilon^2 - \alpha\varepsilon)(u,v) &\ge \frac{\varepsilon^2 - \alpha\varepsilon}{\lambda_1}\|\Delta u\|\|v\| \ge -\frac{\varepsilon\alpha^2}{\lambda_1^2}\|v\|^2 - \frac{\varepsilon}{4}\|\Delta u\|^2\\ &\ge -\frac{\varepsilon}{4}\|\Delta u\|^2 - \frac{\alpha}{2}\|v\|^2\end{aligned} \quad (2.10)$$

and

$$\beta\varepsilon(\Delta u, v) = \beta\varepsilon(\Delta u, u_t + \varepsilon u) = -\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 - \beta\varepsilon^2\|\nabla u\|^2 \quad (2.11)$$

$$(\Delta^2 u, v) = (\Delta u, \Delta v) = (\Delta u, \Delta u_t + \varepsilon\Delta u) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 + \varepsilon\|\Delta u\|^2 \quad (2.12)$$

$$\begin{aligned}&\left(div\left(g\left(|\nabla u|^2\right)\nabla u\right), v\right) = -\left(g\left(|\nabla u|^2\right)\nabla u, \nabla u_t + \varepsilon\nabla u\right)\\ &= -\int_\Omega g\left(|\nabla u|^2\right)\nabla u\nabla u_t\mathrm{d}x - \varepsilon\left(g\left(|\nabla u|^2\right)\nabla u, \nabla u\right)\\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega G\left(|\nabla u|^2\right)\mathrm{d}x - \varepsilon\left(g\left(|\nabla u|^2\right)\nabla u, \nabla u\right).\end{aligned} \quad (2.13)$$

By (2.9)-(2.13), it follows from that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2 + \|\Delta u\|^2 - \beta\varepsilon\|\nabla u\|^2 + \int_\Omega\left(G\left(|\nabla u|^2\right) + C_\eta\right)\mathrm{d}x\right]\\ &+\frac{\alpha}{4}\|v\|^2 + \frac{3\varepsilon}{4}\|\Delta u\|^2 - \beta\varepsilon^2\|\nabla u\|^2 + \varepsilon\left(g\left(|\nabla u|^2\right)\nabla u, \nabla u\right) + \beta\|\nabla v\|^2\\ &\le (\Delta h(u), v) + (f(x), v).\end{aligned} \quad (2.14)$$

By (2.6), we can obtain

$$\begin{aligned}\varepsilon\left(g\left(|\nabla u|^2\right)\nabla u, \nabla u\right) &= \varepsilon\int_\Omega g\left(|\nabla u|^2\right)|\nabla u|^2\,\mathrm{d}x \ge \varepsilon\int_\Omega\left(\rho G\left(|\nabla u|^2\right) - \tilde{C}_\eta\right)\mathrm{d}x\\ &= \varepsilon\rho\int_\Omega\left(G\left(|\nabla u|^2\right) + C_\eta\right)\mathrm{d}x - \varepsilon\rho\int_\Omega C_\eta\mathrm{d}x - \varepsilon\int_\Omega\tilde{C}_\eta\mathrm{d}x.\end{aligned} \quad (2.15)$$

Substituting (2.15) into (2.14), we receive

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2+\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x\right]\\
&+\frac{\alpha}{4}\|v\|^2+\frac{3\varepsilon}{4}\|\Delta u\|^2-\beta\varepsilon^2\|\nabla u\|^2+\varepsilon\rho\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x)+\beta\|\nabla v\|^2\\
&\le\left(\Delta h(u),v\right)+\left(f(x),v\right)+\varepsilon\int_\Omega\tilde{C}_\eta\mathrm{d}x+\varepsilon\rho\int_\Omega C_\eta\mathrm{d}x.
\end{aligned}\tag{2.16}$$

By using Holder inequality, Young's inequality, and (H_2), we obtain

$$\left(f(x),v\right)\le\|f\|\cdot\|v\|\le\frac{2}{\alpha}\|f\|^2+\frac{\alpha}{8}\|v\|^2\tag{2.17}$$

$$\begin{aligned}
\left|\left(\Delta h(u),v\right)\right|&=\left|\left(\nabla h(u),\nabla v\right)\right|\le\int_\Omega|h'(u)||\nabla u||\nabla v|\mathrm{d}x\\
&\le\|h'(u)\|_\infty\cdot\|\nabla u\|\cdot\|\nabla v\|\le\frac{\|h'(u)\|_\infty^2}{2\beta\varepsilon^2}\|\nabla v\|^2+\frac{\beta\varepsilon^2}{2}\|\nabla u\|^2.
\end{aligned}\tag{2.18}$$

Then, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2+\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x\right]+\frac{\alpha}{4}\|v\|^2+\frac{3\varepsilon}{2}\|\Delta u\|^2\\
&-3\beta\varepsilon^2\|\nabla u\|^2+2\varepsilon\rho\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x+2\left(\beta-\frac{\|h'(u)\|_\infty^2}{2\beta\varepsilon^2}\right)\|\nabla v\|^2\\
&\le\frac{4}{\alpha}\|f\|^2+2\varepsilon\int_\Omega\tilde{C}_\eta\mathrm{d}x+2\varepsilon\rho\int_\Omega C_\eta\mathrm{d}x.
\end{aligned}\tag{2.19}$$

Because of $0<\varepsilon<\frac{\lambda_1}{4\beta}$, we get

$$\frac{3}{2}\|\Delta u\|^2-3\beta\varepsilon\|\nabla u\|^2\ge\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2.\tag{2.20}$$

Substituting (2.20) into (2.19) gets

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2+\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x\right]\\
&+\frac{\alpha}{4}\|v\|^2+\varepsilon\|\Delta u\|^2-\beta\varepsilon^2\|\nabla u\|^2+2\varepsilon\rho\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x\\
&+2\left(\beta-\frac{\|h'(u)\|_\infty^2}{2\beta\varepsilon^2}\right)\|\nabla v\|^2\le\frac{4}{\alpha}\|f\|^2+2\varepsilon\int_\Omega\tilde{C}_\eta\mathrm{d}x+2\varepsilon\rho\int_\Omega C_\eta\mathrm{d}x.
\end{aligned}\tag{2.21}$$

Taking $\alpha_1=\min\left\{\frac{\alpha}{4},\varepsilon,2\varepsilon\rho\right\}=\min\{\varepsilon,2\varepsilon\rho\}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}H_1(t)+\alpha_1H_1(t)\le\frac{4}{\alpha}\|f\|^2+2\varepsilon\int_\Omega\tilde{C}_\eta\mathrm{d}x+2\varepsilon\rho\int_\Omega C_\eta\mathrm{d}x:=C_1\tag{2.22}$$

where $H_1(t)=\|v\|^2+\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2+\int_\Omega\left(G\left(|\nabla u|^2\right)+C_\eta\right)\mathrm{d}x$, by using Gronwall inequality,we obtain

$$H_1(t)\le H_1(0)\mathrm{e}^{-\alpha_1t}+\frac{C_1}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1t}\right).\tag{2.23}$$

From (H_1): $|g'(s)|\le C\left(1+|s|^{\frac{m-1}{2}}\right)$, and $1\le m<\infty$ as $N=2$; $1\le m\le m^*\equiv\frac{6-N}{N-2}$ as $3\le N\le 4$; $m=1$

as $N \geq 5$, we have $\int_{\Omega} G\left(|\nabla u|^2\right) \mathrm{d}x \leq C \|\nabla u\|^{m+3}$, according to Embedding Theorem, then $H_0^1 \to L_{m+3}$, let $k = \min\left\{1, \left(1 - \frac{\beta\varepsilon}{\lambda_1}\right)\right\} = 1 - \frac{\beta\varepsilon}{\lambda_1} > 0$, then we have

$$\|(u,v)\|_{V_2 \times H}^2 = \|\Delta u\|^2 + \|v\|^2 \leq \frac{H_1(0)}{k} \mathrm{e}^{-\alpha_1 t} + \frac{C_1}{k\alpha_1}\left(1 - \mathrm{e}^{-\alpha_1 t}\right). \tag{2.24}$$

Then

$$\overline{\lim_{t\to\infty}} \|(u,v)\|_{V_2 \times H}^2 \leq \frac{C_1}{k\alpha_1}. \tag{2.25}$$

So, there exists E_0 and $t_1 = t_1(\Omega) > 0$, such that

$$(u,v)_{V_2 \times H}^2 = \|\Delta u\|^2 + \|v\|^2 \leq E_0 \left(t > t_1\right). \tag{2.26}$$

■

Lemma 2. In addition to the assumptions of Lemma 1, if (H_3): $f \in H^1(\Omega)$, $h \in C^2(\Omega)$, then the solution (u,v) of the problem (1.1)-(1.3) satisfies $(u,v) \in H^3 \times H^1$, and

$$\|(u,v)\|_{H^3 \times H^1}^2 = \|\nabla\Delta u\|^2 + \|\nabla v\|^2 \leq \frac{H_2(0)}{k_2} \mathrm{e}^{-\alpha_2 t} + \frac{C_9}{k_2\alpha_2}\left(1 - \mathrm{e}^{-\alpha_2 t}\right) \tag{2.27}$$

where $v = u_t + \varepsilon u$, $0 < \varepsilon < \min\left\{\frac{\alpha}{4}, \frac{\lambda_1^2}{4\alpha}, \frac{\lambda_1}{4\beta}\right\}$, and $H_2(0) = \|\nabla v_0\|^2 + \|\nabla\Delta u_0\|^2 - \beta\varepsilon \cdot \|\Delta u_0\|^2$, thus there exists E_1 and $t_2 = t_2(\Omega) > 0$, such that

$$\|(u,v)\|_{H^3 \times H^1}^2 = \|\nabla\Delta u\|^2 + \|\nabla v\|^2 \leq E_1\left(t > t_2\right). \tag{2.28}$$

Proof. Taking *H*-inner product by $-\Delta v = -\Delta u_t - \varepsilon\Delta u$ in (2.7), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2 + (\alpha - \varepsilon)\|\nabla v\|^2 + \left(\varepsilon^2 - \alpha\varepsilon\right)(u, -\Delta v) + \beta\varepsilon(\Delta u, -\Delta v) + \left(\Delta^2, -\Delta v\right) + \beta\|\Delta v\|^2 \\ &= \left(div\left(g\left(|\nabla u|^2\right)\nabla u\right), -\Delta v\right) + \left(\Delta h(u), -\Delta v\right) + \left(f(x), -\Delta v\right). \end{aligned} \tag{2.29}$$

Using Holder inequality, Young's inequality and Poincare inequality, we deal with the terms in (2.29) one by one as follow,

$$(\alpha - \varepsilon)\|\nabla v\|^2 \geq \frac{3\alpha}{4}\|\nabla v\|^2 \tag{2.30}$$

$$\begin{aligned} \left(\varepsilon^2 - \alpha\varepsilon\right)(u, -\Delta v) &= \left(\varepsilon^2 - \alpha\varepsilon\right)(\nabla u, \nabla v) \geq \frac{\varepsilon^2 - \alpha\varepsilon}{\lambda_1}\|\nabla\Delta u\|\|\nabla v\| \\ &\geq -\frac{2\varepsilon\alpha^2}{\lambda_1^2}\|\nabla v\|^2 - \frac{\varepsilon}{8}\|\nabla\Delta u\|^2 \geq -\frac{\varepsilon}{8}\|\nabla\Delta u\|^2 - \frac{\alpha}{2}\|\nabla v\|^2 \end{aligned} \tag{2.31}$$

and

$$\beta\varepsilon(\Delta u, -\Delta v) = \beta\varepsilon\left(\Delta u, -\Delta u_t - \varepsilon\Delta u\right) = -\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2 - \beta\varepsilon^2\|\Delta u\|^2 \tag{2.32}$$

$$\left(\Delta^2 u, -\Delta v\right) = (\nabla\Delta u, \nabla\Delta v) = \left(\nabla\Delta u, \nabla\Delta u_t + \varepsilon\nabla\Delta u\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\Delta u\|^2 + \varepsilon\|\nabla\Delta u\|^2. \tag{2.33}$$

Substituting (2.30)-(2.33) into (2.29), we can obtain that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\nabla\Delta u\|^2-\beta\varepsilon\|\Delta u\|^2\right)+\frac{\alpha}{4}\|\nabla v\|^2+\frac{7\varepsilon}{8}\|\nabla\Delta u\|^2-\beta\varepsilon^2\|\Delta u\|^2+\beta\|\Delta v\|^2\\&\le\left(div\left(g\left(|\nabla u|^2\right)\nabla u\right),-\Delta v\right)+\left(\Delta h(u),-\Delta v\right)+\left(f(x),-\Delta v\right).\end{aligned}\tag{2.34}$$

By using Holder inequality, Young's inequality, and (H_1), (H_3), we obtain

$$\left(f(x),-\Delta v\right)=\left(\nabla f(x),\nabla v\right)\le\|\nabla f\|\cdot\|\nabla v\|\le\frac{2}{\alpha}\|\nabla f\|^2+\frac{\alpha}{8}\|\nabla v\|^2\tag{2.35}$$

$$\begin{aligned}\left|\left(\Delta h(u),-\Delta v\right)\right|&=\left|\left(\nabla\cdot\left(h'(u)\nabla u\right),\Delta v\right)\right|=\left|\left(h''(u)|\nabla u|^2+h'(u)\Delta u,\Delta v\right)\right|\\&\le\left|\left(h''(u)|\nabla u|^2,\Delta v\right)\right|+\left|\left(h'(u)\Delta u,\Delta v\right)\right|\\&\le\|h''(u)\|_\infty\cdot\|\nabla u\|_4^2\cdot\|\Delta v\|+\|h'(u)\|_\infty\cdot\|\nabla u\|\cdot\|\Delta v\|\\&\le\frac{\beta}{4}\|\Delta v\|^2+\frac{2\|h''(u)\|_\infty^2\cdot\|\nabla u\|_4^4}{\beta}+\frac{2\|h'(u)\|_\infty^2\|\nabla u\|^2}{\beta}.\end{aligned}\tag{2.36}$$

By using Gagliardo-Nirenberg inequality, and according the Lemma 1, we can get $\|\nabla u\|_4\le C_2\|\Delta u\|^{\frac{1}{4n}}\|\nabla u\|^{\frac{4n-1}{4n}}:=C_3$. Then, we have

$$\left|\left(\Delta h(u),-\Delta v\right)\right|\le\frac{\beta}{4}\|\Delta v\|^2+C_4\left(\|h''(u)\|_\infty,\|h'(u)\|_\infty,\|\nabla u\|,C_3,\beta\right).\tag{2.37}$$

By using the same inequality, we can obtain

$$\begin{aligned}&\left|\left(div\left(g\left(|\nabla u|^2\right)\nabla u\right),-\Delta v\right)\right|\\&=\left|\left[2g'\left(|\nabla u|^2\right)|\nabla u|^2+g\left(|\nabla u|^2\right)\right]\Delta u,\Delta v\right|\le\left|C_4\left(1+|\nabla u|^{m+1}\right)\Delta u,\Delta v\right|\\&\le\left|\left(C_4\Delta u,\Delta v\right)\right|+C_4\left|\left(|\nabla u|^{m+1}\Delta u,\Delta v\right)\right|\\&\le C_4\|\Delta u\|\cdot\|\Delta v\|+C_4\|\nabla u\|_{4(m+1)}^{m+1}\cdot\|\Delta u\|_4\cdot\|\Delta v\|\\&\le\frac{\beta}{4}\|\Delta v\|^2+\frac{2}{\beta}C_4^2\|\Delta u\|^2+\frac{2}{\beta}C_4^2\|\nabla u\|_{4(m+1)}^{2(m+1)}\cdot\|\Delta u\|_4^2.\end{aligned}\tag{2.38}$$

By using Gagliardo-Nirenberg inequality, and according the Lemma 1, we can get $\|\nabla u\|_{4(m+1)}\le C_5\|\Delta u\|^{\frac{2m+1}{4(m+1)n}}\|\nabla u\|^{\frac{4(m+1)n-2m-1}{4(m+1)n}}:=C_6$, $\|\Delta u\|_4\le C_7\|\nabla\Delta u\|^{\frac{1}{4n}}\|\Delta u\|^{\frac{4n-1}{4n}}$. Then, by using Young's inequality, we have

$$\begin{aligned}&\left|\left(div\left(g\left(|\nabla u|^2\right)\nabla u\right),-\Delta v\right)\right|\le\frac{\beta}{4}\|\Delta v\|^2+\frac{\epsilon^{4n}}{4n}\left(\|\nabla\Delta u\|^{\frac{1}{2n}}\right)^{4n}\\&+\frac{4n-1}{4n}\epsilon^{\frac{1-4n}{4n}}\left(\frac{2}{\beta}C_4^2C_6^{2m+2}C_7^2\|\Delta u\|^{\frac{4n-1}{2n}}\right)^{\frac{4n}{4n-1}}+\frac{2}{\beta}C_4^2\|\Delta u\|^2\end{aligned}\tag{2.39}$$

where $\epsilon^{4n}=\frac{n\varepsilon}{2}$, then

$$\left|\left(div\left(g\left(|\nabla u|^2\right)\nabla u\right),-\Delta v\right)\right|\le\frac{\beta}{4}\|\Delta v\|^2+\frac{\varepsilon}{8}\|\nabla\Delta u\|^2+C_8\left(n,\varepsilon,\beta,C_4,C_6,C_7,\|\Delta u\|\right).\tag{2.40}$$

Substituting (2.35), (2.37), (2.40) into (2.34), we receive

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|^2+\|\nabla\Delta u\|^2-\beta\varepsilon\|\Delta u\|^2\right)+\frac{\alpha}{4}\|\nabla v\|^2+\frac{3\varepsilon}{2}\|\nabla\Delta u\|^2-2\beta\varepsilon^2\|\Delta u\|^2+\beta\|\Delta v\|^2\\&\leq 2\left(\frac{2}{\alpha}\|\nabla f\|^2+C_4+C_8\right).\end{aligned} \tag{2.41}$$

Because of $0<\varepsilon<\frac{\lambda_1}{4\beta}$, we get

$$\frac{3}{2}\|\Delta u\|^2-2\beta\varepsilon\|\nabla u\|^2\geq\|\Delta u\|^2-\beta\varepsilon\|\nabla u\|^2. \tag{2.42}$$

Taking $\alpha_2=\min\left\{\frac{\alpha}{4},\varepsilon\right\}=\varepsilon$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}H_2(t)+\alpha_2H_2(t)\leq 2\left(\frac{2}{\alpha}\|\nabla f\|^2+C_4+C_8\right):=C_9 \tag{2.43}$$

where $H_2(t)=\|\nabla v\|^2+\|\nabla\Delta u\|^2-\beta\varepsilon\|\Delta u\|^2$, by Gronwall inequality, we have

$$H_2(t)\leq H_2(0)\mathrm{e}^{-\alpha_2t}+\frac{C_9}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2t}\right). \tag{2.44}$$

Let $k_2=\min\left\{1,\left(1-\frac{\beta\varepsilon}{\lambda_1}\right)\right\}=1-\frac{\beta\varepsilon}{\lambda_1}>0$, so we get

$$\|(u,v)\|^2_{H^3\times H^1}=\|\nabla\Delta u\|^2+\|\nabla v\|^2\leq\frac{H_2(0)}{k_2}\mathrm{e}^{-\alpha_2t}+\frac{C_9}{k_2\alpha_2}\left(1-\mathrm{e}^{-\alpha_2t}\right). \tag{2.45}$$

Then

$$\varlimsup_{t\to\infty}\|(u,v)\|^2_{H^3\times H^1}\leq\frac{C_9}{k_2\alpha_2}. \tag{2.46}$$

So, there exists E_1 and $t_2=t_2(\Omega)>0$, such that

$$\|(u,v)\|^2_{H^3\times H^1}=\|\nabla\Delta u\|^2+\|\nabla v\|^2\leq E_1\ (t>t_2). \tag{2.47}$$

■

3. Global Attractor

3.1. The Existence and Uniqueness of Solution

Theorem 3.1. *Assume that* $(H_1)\ g\in C^1(\Omega)$,

$$\liminf_{|s|\to\infty}\frac{G(s)}{|s|^{\frac{m+3}{2}}}\geq -C$$

$$\liminf_{|s|\to\infty}\frac{sg(s)-\rho G(s)}{|s|^{\frac{m+3}{2}}}\geq -C$$

where $G(s)=\int_0^s g(\tau)\mathrm{d}\tau$, $0<\rho<2$ and $\lambda_1(>0)$ is the first eigenvalue of the $-\Delta$, and when $N\geq 2$,

$$|g'(s)|\leq C\left(1+|s|^{\frac{m-1}{2}}\right),\quad s\in\Omega,$$

where $1 \le m < \infty$ as $N = 2$; $1 \le m \le m^* \equiv \dfrac{6-N}{N-2}$ as $3 \le N \le 4$; $m = 1$ as $N \ge 5$.

$(H_2)(u_0, u_1) \in H^3 \times H^1$, $f \in H$, $h \in C^2$ and $\|h'(u)\|_\infty < \dfrac{\sqrt{2}\lambda_1}{4}$.

Then the problem (1.1)-(1.3) exists a unique smooth solution

$$(u, u_t) \in L^\infty\left([0, +\infty); H^3(\Omega) \times H^1(\Omega)\right).$$

Remark 2. We denote the solution in Theorem 3.1 by $S(t)(u_0, u_1) = (u(t), u_t(t))$. Then $S(t)$ composes a continuous semigroup in $H^3 \times H^1$.

Proof of Theorem 3.1.

Proof. By the Galerkin method and Lemma 1, we can easily obtain the existence of Solutions. Next, we prove the uniqueness of Solutions in detail. Assume u, v are two solutions of (1.1)-(1.3), let $w = u - v$, then $w(x,0) = w_0(x) = 0, w_t(x,0) = w_1(x) = 0$ and the two equations subtract and obtain

$$w_{tt} + \alpha w_t - \beta \Delta w_t + \Delta^2 w = div\left[g\left(|\nabla u|^2\right)\nabla u - g\left(|\nabla v|^2\right)\nabla v\right] + \Delta\left(h(u) - h(v)\right). \tag{3.1}$$

Taking H-inner product by w_t in (3.1), we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2 + \|\Delta w\|^2\right) + \alpha\|w_t\|^2 + \beta\|\nabla w_t\|^2 \\ &= \left(div\left[g\left(|\nabla u|^2\right)\nabla u - g\left(|\nabla v|^2\right)\nabla v\right], w_t\right) + \left(\Delta\left(h(u) - h(v)\right), w_t\right). \end{aligned} \tag{3.2}$$

By (H_1), (H_2)

$$\begin{aligned} \left|\left(\Delta\left(h(u) - h(v)\right), w_t\right)\right| &= \left|\left(h(u) - h(v), \Delta w_t\right)\right| = \left|\left(h'(\xi)w, \Delta w_t\right)\right| \\ &\le \|h'(\xi)\|_\infty \cdot \|\Delta w\| \cdot \|w_t\| \le \alpha\|w_t\|^2 + \frac{\left(\|h'(\xi)\|_\infty\right)^2}{4\alpha}\|\Delta w\|^2 \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\left|\left(div\left[g\left(|\nabla u|^2\right)\nabla u - g\left(|\nabla v|^2\right)\nabla v\right], w_t\right)\right| = \left|\left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\left(g\left(|\nabla U_\theta|^2\right)\nabla U_\theta\right)\mathrm{d}\theta, w_t\right)\right| \\ &= \left|\left(\int_0^1\left(2g'\left(|\nabla U_\theta|^2\right)|\nabla U_\theta|^2 + g\left(|\nabla U_\theta|^2\right)\right)\mathrm{d}\theta \nabla w, \nabla w_t\right)\right| \\ &\le C_{10}\left|\left(\int_0^1\left(1 + |\nabla U_\theta|^{m+1}\right)\mathrm{d}\theta \nabla w, \nabla w_t\right)\right| \\ &\le C_{10}\left|\left(\nabla w, \nabla w_t\right)\right| + C_{10}\left|\left(\int_0^1 |\nabla U_\theta|^{m+1}\mathrm{d}\theta \nabla w, \nabla w_t\right)\right| \\ &\le \alpha\|w_t\|^2 + \frac{C_{10}^2}{4\alpha}\|\Delta w\|^2 + C_{10}\int_0^1 \|\nabla U_\theta\|_{4(m+1)}^{m+1}\mathrm{d}\theta \|\nabla w\|_4 \cdot \|\nabla w_t\| \\ &\le \alpha\|w_t\|^2 + \frac{\beta}{2}\|\nabla w_t\|^2 + \left[\frac{C_{10}^2}{4\alpha} + \frac{C_{10}^2}{2\beta}\left(\int_0^1 |\nabla U_\theta|^{m+1}\mathrm{d}\theta\right)^2\right] \cdot \|\Delta w\|^2 \end{aligned} \tag{3.4}$$

where $\min\{u, v\} \le \xi \le \max\{u, v\}, U_\theta = \theta u + (1-\theta)v, 0 < \theta < 1$.

By using Gagliardo-Nirenberg inequality, and according the Lemma 1,we can get

$\|\nabla U_\theta\|_{4(m+1)} \le C_{11}\|\Delta U_\theta\|^{\frac{2m+1}{4(m+1)n}} \|\nabla U_\theta\|^{\frac{4(m+1)n-2m-1}{4(m+1)n}} := C_{12}$. Then, we have

$$\left|\left(div\left[g\left(|\nabla u|^2\right)\nabla u - g\left(|\nabla v|^2\right)\nabla v\right], w_t\right)\right| \le \alpha\|w_t\|^2 + C_{13}\left(C_{10}, C_{12}, \beta, \alpha\right) \cdot \|\Delta w\|^2. \tag{3.5}$$

Substituting (3.3), (3.5) into (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\|\Delta w\|^2\right)+\beta\|\nabla w_t\|^2 \le 2\left[\frac{\left(\|h'(\xi)\|_\infty\right)^2}{4\alpha}+C_{13}\right]\|\Delta w\|^2+2\alpha\|w_t\|^2. \quad (3.6)$$

Taking $B=\max\left\{2\left[\frac{\left(\|h'(\xi)\|_\infty\right)^2}{4\alpha}+C_{13}\right],2\alpha\right\}$.

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+\|\Delta w\|^2\right)\le B\left(\|\Delta w\|^2+\|w_t\|^2\right). \quad (3.7)$$

By using Gronwall inequality, we obtain

$$\|w_t\|^2+\|\Delta w\|^2 \le \left(\|\Delta w(0)\|^2+\|w_t(0)\|^2\right)\mathrm{e}^{Bt}. \quad (3.8)$$

So, we can get $\|w_t\|^2+\|\Delta w\|^2\le 0$ because of $w_0(x)=0, w_1(x)=0$.
That shows that

$$\|w_t\|^2=0,\quad \|\Delta w\|^2=0.$$

That is

$$w(x,t)=0.$$

Therefore

$$u=v.$$

We get the uniqueness of the solution. So the proof of the Theorem 3.1. has been completed. ■

3.2. Global Attractor

Theorem 3.2. [10] Let X be a Banach space, and $\{S(t)\}(t\ge 0)$ are the semigroup operator on X. $S(t):X\to X, S(t+s)=S(t)S(s)(\forall t,s\ge 0), S(0)=I$, here I is a unit operator. Set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is bounded, namely $\forall R>0, \|u\|_X\le R$, it exists a constant $C(R)$, so that

$$\|S(t)u\|_X\le C(R)(t\in[0,+\infty));$$

2) It exists a bounded absorbing set $B_0\subset X$, namely, $\forall B\subset X$, it exists a constant t_0, so that

$$S(t)B\subset B_0(t\ge t_0);$$

here B_0 and B are bounded sets.

3) When $t>0$, $S(t)$ is a completely continuous operator.

Therefore, the semigroup operators $S(t)$ exist a compact global attractor A.

Theorem 3.3 Under the assume of Theorem 3.1, equations have global attractor

$$A=\omega(B_0)=\bigcap_{s\ge 0}\overline{\bigcup_{t\ge s}S(t)B_0},$$

where $B_0=\left\{(u,v)\in H^3\times H^1:\|(u,v)\|^2_{H^3\times H^1}=\|u\|^2_{H^3}+\|v\|^2_{H^1}\le E_0+E_1\right\}$, B is the bounded absorbing set of $H^3\times H^1$ and satisfies

1) $S(t)A=A, t>0$;

2) $\lim\limits_{t\to\infty} dist\left(S(t)B,A\right)=0$, here $B\subset H^3\times H^1$ and it is a bounded set,

$dist\left(S(t)B, A\right) = \sup_{x\in B,}\inf_{y\in A}\left\|S(t)x - y\right\|_{H^3\times H^1}$.

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup $S(t)$, here $X = H^3\times H^1$, $S(t): H^3\times H^1 \to H^3\times H^1$.

(1) From Lemma 1-Lemma 2, we can ge that $\forall B\subset H^3\times H^1$ is a bounded set that includes in the ball $\left\{\left\|(u,v)\right\|_{H^3\times H^1}\le R\right\}$,

$$\left\|S(t)(u_0,v_0)\right\|^2_{H^3\times H^1} = \|u\|^2_{H^3} + \|v\|^2_{H^1} \le \|u_0\|^2_{H^3} + \|v_0\|^2_{H^1} + C \le R^2 + C, \left(t\ge 0, (u_0,v_0)\in B\right)$$

This shows that $S(t)(t\ge 0)$ is uniformly bounded in $H^3\times H^1$.

(2) Furthermore, for any $(u_0,v_0)\in H^3\times H^1$, when $t\ge \max\{t_1,t_2\}$, we have

$$\left\|S(t)(u_0,v_0)\right\|^2_{H^3\times H^1} = \|u\|^2_{H^3} + \|v\|^2_{H^1} \le E_0 + E_1$$

So we get B_0 is the bounded absorbing set.

(3) Since $H^3\times H^1 \to V_2\times H$ is compact embedded, which means that the bounded set in $V_3\times H^1$ is the compact set in $V_2\times H$, so the semigroup operator $S(t)$ exist a compact global attractor A. Theorem 3.3 is proved. ■

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