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Optimal Stochastic Pine Stands Harvest Rotation Policies

Eduardo Navarrete

Department of System Engineering, Universidad de la Frontera, Temuco, Chile
Email: eduardo.navarrete@ufrontera.cl

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Abstract

A new Faustmann optimal rotation harvesting stands' problem under Brown geometric price and Logistic and Gompertz wood stock, diffusions is presented. The optimal cut policies for the stochastic Faustmann model and the single harvest rotation or Vicksell model are evaluated in the case of a Chilean Radiata pine forest company. The company cut policy validates the Vicksell model, its optimal cut policies overestimate the company policy cut in 1.2%, in the Gompertz case, and underestimate it in 2.3%, in the Logistic case. The Faustmann optimal cut policies present a larger underestimation of the company cut policy in 10.1%, in the Gompertz case, and in 21.5%, in the Logistic case. The preference for shorter evaluation period that the company shows is due to the organizational risk that the forest economic sectors has in Chile.

Keywords

Optimal Tree Cutting, Faustmann Stochastic Formula Component, Optimal Stopping Problem

1. Introduction

The optimal forest harvest models development is marked by the emergence of two controversial issues. The first issue was formulated in its early beginning by Faustmann in 1849. He formulated a multiple rotation version considering the dynamic effect that the future renovations had in the rotation period of the cut and planting of the trees. The issue was not considered relevant by researchers and forester management, who preferred a simple rotation or Vicksell model. The Faustmann model was finally rediscovered, first by Gaffney in 1957, and finally by Samuelson in 1976, who validated Faustmann's deterministic formula as the correct one, since it was the only one to consider the cost of land rent. An increasing number of researchers continued extending this model (Brazee, 2001; Chang, 2001; Amacher et al., 2011). The second issue was strongly formulated by Samuelson (1976), calling to replace the simple notion of stationary equilibrium by the notion of perpetual Brownian motion of wood price. Many researchers followed his recommendation and considered the single rotation prob-

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lem as an American call option, taking into account wood price as a geometric Brown diffusion (see, Newman, 2002; Clarke & Reed, 1989; Thomson, 1992; Platinga et al., 1998). Other authors also considered the wood stock as a second geometric Brown diffusion (Morck & Schwartz, 1989; Insley, 2002), or as a Logistic diffusion (Alvarez & Koskela, 2007; Navarrete, 2011). Willassen (1998) introduced uncertainty to the Faustmann model, incorporating the forest growth as a stochastic Markov diffusion process and characterizing the properties of the optimal solutions. Sodal (2002) simplified Willassen's approach in a closed-form rotation formula for the same state stochastic variable and Insley & Rollins (2005) proposed a numeric algorithm to solve its Hamilton-Jacobi-Bellman solution. Navarrete & Bustos (2013) extended the Faustmann model considering, price as a geometric Brown and wood stock as a Logistical or Gompertz, diffusions processes, transforming the model into an equivalent optimal stopping problem, in spite of its contribution. This paper requires some improvement which will be done in the present paper. Therefore this paper has two basic objectives. In section 2.3, the proof of the equivalence of the Faustmann rotation model under Brown Price and ITO wood stock independent diffusion processes and a one dimensional modified wood stock Optimal Stopping problem is formalized as a reformulation lemma. In section 4, a new algorithm is developed to obtain the Faustmann equivalent one dimensional optimal stopping solution, intersecting the optimal condition curve with the expected wood stock growth feasibility restriction. Finally new stands growth data for the middle age 11 and 12 years is incorporated in Section 3, in order to validate Vicksell and Faustmann stands cut models, under price and wood stock uncertainty.

2. Stochastic Harvest Models

2.1. Stochastic Harvest Stands Rotation Models

The basic model considers two independent stochastic processes: an ITO diffusion for the wood stock, Equation (1) and a geometric Brownian diffusion, Equation (2). The functional objective for the single rotation is given by Equation (3) and for the multiple rotations by Equation (4) see Johnson (2006):

$$\mathrm{d}V_t = \mu(V_t)\mathrm{d}t + \sigma(V_t)\mathrm{d}W_V \tag{1}$$

$$\mathrm{d}P_t = \alpha P_t \mathrm{d}t + \beta P_t \mathrm{d}W_P \tag{2}$$

$$F^s(\bar{V},T) = \sup_{\forall(t\geq 0)}\left[E^R_{(V,P)}\left(\mathrm{e}^{-it}P_tV_t - C\right)\right] \tag{3}$$

$$F^M(\bar{V},T) = \sup_{\forall(t\geq 0)}\left[E^R_{(V,P)}\frac{\left(\mathrm{e}^{-it}P_tV_t - C\right)}{\left(1-\mathrm{e}^{-it}\right)}\right] \tag{4}$$

Model notation:

Deterministic state variable	*Economic Parameter*
t = Wood stock age	C = Stand regeneration cost
Stochastic sate variables	P_0 = Price at time 0
V_t = Wood stock	$c = C/P_0$
P_t = Wood stumpage price at time t	R, Q = Probabilistic metrics
Diffusion Parameters	F, Z = Functional objective
W_v = Wiener wood stock	i = Risky rate of return
$\mu(V)$ = Wood stock diffusion drift rate parameter	$r_T = i/\left(1-\mathrm{e}^{-iT}\right)$, Capitalized rate of return
$\sigma(V)$ = Wood stock volatility parameter	*Optimal Parameters*
W_p = Wiener price	T = Optimal cut time
α = Wood price diffusion drift rate	$\bar{V}$ = Expected optimal cut volume
β = Wood price volatility	

2.2. Reformulation of the Simple Harvest Rotation Problem

The stochastic model (1) (2) and (3) is difficult to solve due to both diffusions. But it is equivalent to the following one dimensional diffusion Optimal Stopping problem, (5) and (6) see Navarrete (2011).

$$Z^{S}\left(\bar{V},T\right)=\sup_{\forall(t\geq 0)}\left[E^{Q}_{(V_t)}\left(\mathrm{e}^{-(i-\alpha)t}V_t-c\right)\right] \tag{5}$$

$$\mathrm{d}V_t=\left[\mu\left(V_t\right)+\beta\sigma\left(V_t\right)\right]\mathrm{d}t+\sigma\left(V_t\right)\mathrm{d}\bar{W}_V \tag{6}$$

The solution of this problem is given by the (HJB) Equation (7), see Navarrete (2011).

$$\operatorname*{Max}_{\bar{V}\geq 0}\left\{\left(\frac{1}{2}\right)\sigma^2\bar{V}^2F''\left(\bar{V}\right)+\left[\mu\left(\bar{V}\right)+\beta\sigma\left(\bar{V}\right)\right]F'\left(\bar{V}\right)-(i-\alpha)F\left(\bar{V}\right),\bar{V}-F\left(\bar{V}\right)\right\}=0 \tag{7}$$

Assuming the existence of a frontier V^* that divides the zone into a continuation zone (no-cutting) and a stopping zone (immediate-cutting), the solution to the equation HJB is given by (8) for the continuation zone (8) and stopping zone (9). A solution of (8) is given by (10) (see Johnson, 2006).

$$\left(\frac{1}{2}\right)\sigma^2\bar{V}^2F''\left(\bar{V}\right)+\left[\mu\left(\bar{V}\right)+\beta\sigma\left(\bar{V}\right)\right]F'\left(\bar{V}\right)-(i-\alpha)F\left(\bar{V}\right)=0 \tag{8}$$

$$\bar{V}-F\left(\bar{V}\right)=0 \tag{9}$$

$$F\left(\bar{V}\right)=\begin{cases}A\Psi\left(\bar{V}\right)+B\Phi\left(\bar{V}\right) & \bar{V}<V^*\\ \bar{V} & \bar{V}\geq V^*\end{cases} \tag{10}$$

where Ψ (resp., Φ) is strictly increasing (resp., decreasing), function since the payoff function are bounded and small, V is positive and should remain bounded and positive a $V\to 0$, necessarily then $B\to 0$. The solution also fulfils the smooth-pasting condition at the free boundary point V^*, equations (11) and eliminating the constant A gives Equation (12).

$$A\Psi\left(\bar{V}\right)=\bar{V}\ \text{ and }\ A\Psi'\left(\bar{V}\right)=1 \tag{11}$$

$$\Psi\left(\bar{V}\right)=\bar{V}\Psi'\left(\bar{V}\right). \tag{12}$$

2.3. Reformulation of the Multiple Harvest Rotation Problems

The Faustmann optimal rotation model (1) (2) and (4) its equivalent to the following Optimal Stopping one dimensional diffusion problem (13) and (14), see **Lemma 1**.

***Lemma* 1**. Reformulation Lemma:

The Faustmann optimal rotation problem is equivalent to the following Optimal Stopping one dimensional diffusion problem (13) and (14).

$$Z^{M}\left(\bar{V},T\right)=\sup_{\forall(t\geq 0)}E^{Q}_{V}\frac{\left(\mathrm{e}^{-(i-\alpha)t}V_t-c\right)}{\left(1-\mathrm{e}^{-it}\right)} \tag{13}$$

$$\mathrm{d}V_t=\left\{\mu\left(V_t\right)+\beta\sigma\left(V_t\right)\right\}\mathrm{d}t+\sigma\left(V_t\right)\mathrm{d}\bar{W} \tag{14}$$

Proof:

Given the independence of both variables the expectation can be calculated as the product of two independent expectation "$E^{R}_{(V_t)}E^{R}_{(V_t)}$" that are independent of the deterministic parameters e^{-it} and $1/\left(1-\mathrm{e}^{-it}\right)$ and the constant C, so they can be taken outside its domain. Introducing the integration of the price Brown diffusion $P_t=P_0\mathrm{e}^{\alpha t}M_t$ with $M_t=\exp\left(\beta Wp-1/2\,\beta^2t\right)$ and $C=P_0c$ in the functional objective (4) results in the objective (15) and reduce the Faustmann problem to the following one dimensional diffusion problem (15) and (16).

$$F^{M}\left(\bar{V},T\right)=P_o\sup_{\forall(t\geq 0)}\left\{\frac{1}{\left(1-\mathrm{e}^{-it}\right)}\left[E^{R}_{V_t}\left\{E^{R}_{P_t}\left(\mathrm{e}^{-(i-\alpha t)}M_tV_t-c\right)\right\}\right]\right\} \tag{15}$$

$$\mathrm{d}V_t=\mu\left(V_t\right)\mathrm{d}t+\sigma\left(V_t\right)\mathrm{d}W_V \tag{16}$$

Dividing the objective functional (15) by the constant P_0 and applying the Thijssen version of the Girsanov theorem to Equations (15) and (16), for the Martingale M_t and the Radom-Nykodym derivative $\mathrm{d}R/\mathrm{d}Q = M_t$ (see Thijssen, 2010, **Appendix A**) permits to reformulate the problem (13) (14) by its equivalent Q metric optimal stopping problem (16) and (17).

The formulation of the HJB equation for this problem is given by Equation (17) for the capitalized risky rate of return $r_T = i/\left(1-\mathrm{e}^{-iT}\right)$ see Navarrete & Bustos (2013).

$$\underset{\bar{V}\geq 0}{\mathrm{Max}}\left\{\left(\frac{1}{2}\right)\sigma^2\overline{V^2}F''\left(\bar{V}\right)+\left[\mu\left(\bar{V}\right)+\beta\sigma\left(V_t\right)\right]F'\left(\bar{V}\right)-(r_T-\alpha)F\left(\bar{V}\right)-cr_T,\frac{\left(\bar{V}-c\right)}{1-\mathrm{e}^{-iT}}-F\left(\bar{V}\right)\right\} \tag{17}$$

In this case the differential equation for the continuation region $\left(\bar{V}\leq V^*\right)$ is given by the non homogenous differential Equation (18), and by Equation (19) for the stopping zone $\bar{V}\left(>V^*\right)$.

$$\left(\frac{1}{2}\right)\sigma^2\overline{V^2}F''\left(\bar{V}\right)+\left[\mu\left(\bar{V}\right)+\beta\sigma\left(V_t\right)\right]F'\left(\bar{V}\right)-\left(r_T-\alpha\right)F\left(\bar{V}\right)-cr_T=0;\ \text{with},\ F\left(0\right)=\frac{r_T}{i}c \tag{18}$$

$$\frac{\bar{V}-c}{\left(1-\mathrm{e}^{-iT}\right)}-F\left(\bar{V}\right)=0 \tag{19}$$

The solution to the ordinary differential equation, (18) under the initial condition for a given risky rate of return r_T is given in Equation (20), with $\psi\left(\bar{V}\right)$, the positive increasing solution of the homogenous part and $\left[r_T/i\right]c$ the particular solution of Equation (18).

$$F\left(\bar{V}\right)=\begin{cases}A\Psi\left(\bar{V}\right)-\dfrac{r_T}{i}c & \bar{V}<V^*\\ \bar{V} & \bar{V}\geq V^*\end{cases} \tag{20}$$

In this case the smooth pasting condition for each parameter r_T are given by Equation (21):

$$A\Psi\left(\bar{V}\right)-\frac{r_T}{i}c=\frac{\bar{V}-c}{\left(1-\mathrm{e}^{-iT}\right)}=\frac{r_T}{i}\left(\bar{V}-c\right),\quad A\Psi'\left(\bar{V}\right)=\frac{r_T}{i} \tag{21}$$

So $\bar{V}$ must fulfill smooth-pasting condition (22) for each parameter r_T.

$$\Psi\left(\bar{V}\right)=\bar{V}\Psi'\left(\bar{V}\right) \tag{22}$$

2.4. Wood Stock Sigmoid Diffusion Equations

The basic requirement of a pine stand growing diffusion is its sigmoid pattern (Garcia, 2005). The logistic diffusion, Equation (23) is a special case of the sigmoid model given by $\mu\left(V\right)=\mu V\left(1-\gamma V\right)$ and $\sigma\left(V\right)=\sigma V$, where μ and γ are the drift and saturation parameters and σ is the volatility parameter.

$$\mathrm{d}V_t=\mu V_t\left(1-\gamma V_t\right)\mathrm{d}t+\sigma V_t\mathrm{d}W_V \tag{23}$$

The integration of the value of V is given by Equation (24) (Kloeden & Platen, 1992: p. 125) and its expected value is given by Equation (25)

$$V_t=\frac{V_0\mathrm{Exp}\left[\left(\mu-\dfrac{\sigma^2}{2}\right)t+\sigma W\right]}{1+\mu\gamma V_0\displaystyle\int_{t_m}^{t}\mathrm{Exp}\left[\left(\mu-\dfrac{\sigma^2}{2}\right)s+\sigma W\right]\mathrm{d}s} \tag{24}$$

$$E\left(V_t\right)=\frac{\dfrac{1}{\gamma}}{1+\dfrac{1-\gamma V_0}{\gamma V_0}\mathrm{e}^{-\mu t}} \tag{25}$$

Other important sigmoid diffusion is the Gompertz geometrical diffusion, which is given by Equation (26), which is integrated to the expression (27), and its expected values given by expression (28) (see Gutierrez, 2009).

$$\mathrm{d}V_t = kV_t\left[\theta - \ln\left(V_t\right)\right]\mathrm{d}t + \sigma V_t \mathrm{d}W_V \tag{26}$$

$$V_t = \exp\left\{\ln\left(V_0\right)\mathrm{e}^{-kt} + \left[\theta - \frac{\sigma^2}{2k}\right]\left(1-\mathrm{e}^{-kt}\right) + \sigma\mathrm{e}^{-kt}\int \mathrm{d}W_V\right\} \tag{27}$$

$$E\left(V_t\right) = \exp\left\{\ln\left(V_0\right)\mathrm{e}^{-kt} + \left[\theta - \frac{\sigma^2}{2k}\right]\left(1-\mathrm{e}^{-kt}\right) + \frac{\sigma^2}{4k}\left(1-\mathrm{e}^{-2kt}\right)\right\} \tag{28}$$

The Geometric Brown Price diffusion is given by Equation (29), and integrates in Equation (30).

$$\mathrm{d}P_t = \alpha P_t \mathrm{d}t + \beta P_t \mathrm{d}W_P \tag{29}$$

$$P_t = P_0 \mathrm{e}^{\alpha t}\exp\left(\beta W_P - \frac{1}{2}\beta^2 t\right) \tag{30}$$

3. Experimental Data and Parameter Fitting

3.1. Logistic Diffusion Fitting

The experimental data belonged to 122 harvest pine stands of a Chilean Forest Company in the Araucania Chile region, between years 1999 and 2005 (see **Appendix A**), and came from different sample plots, with site indexes between 30 and 35 meters, representing sites with high forest aptitude and a tree average initial volume of 32 m^3/ha at the first 4 years after initial seed cultivation period. The additional 20 point for years 11 and 12 were taken from Alvarez et al. (2012). The business harvest cut data for a Logistic diffusion for the 95% confidence range is given by data point plotted in **Figure 1**.

The logistic diffusion parameter cannot be adjusted by maximum verisimilitude, see (Beskos et al., 2006), so it was fitted using a logistical nonlinear regression and a Monte Carlo/Bootstrap simulation sampling method, implemented by Meyer et al. (Loglet Lab.1 software, 1999). Choosing $V_0 = 1/2\gamma$ = half saturation volume, and $T_0 = T_m$, time to achieve that volume, Equation (25) is transformed into the more conventional expression (31).

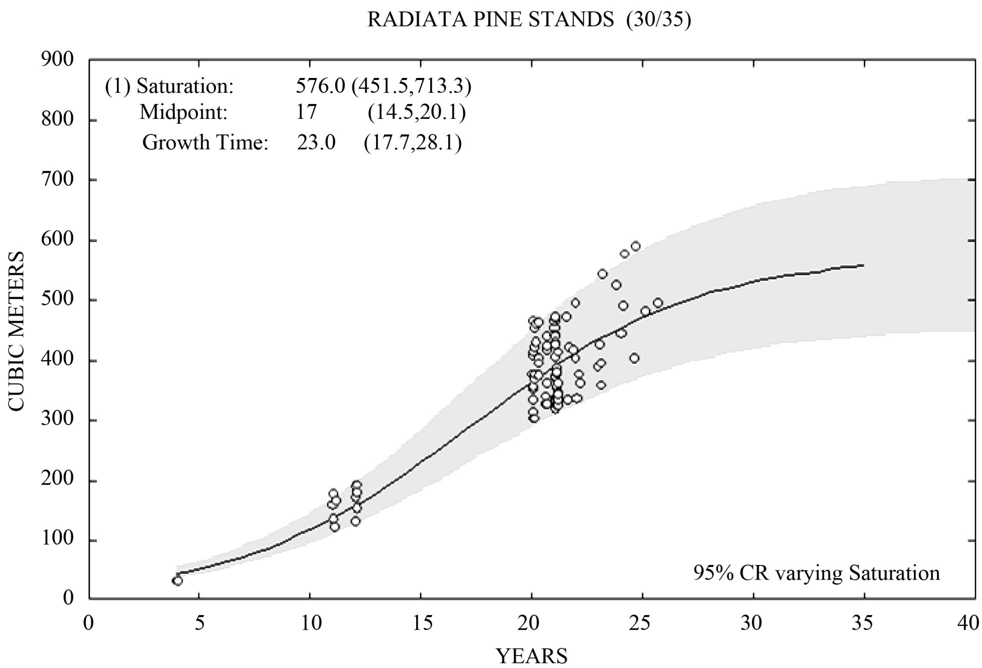

Figure 1. Wood volume per hectare, (m^3/ha) versus years.

$$E(V_t) = \frac{1/\gamma}{1+e^{-\mu(t-t_m)}} \tag{31}$$

With; $1/\gamma$ = saturation volume, μ = growth rate parameter, V_s = Saturation volume and t_m = time to achieve the midpoint of the saturation volume. The standard deviation $Sd(\infty)$, at the saturation zone is constant, and given by Equation (32).

$$Sd(\infty) = \sigma V_s = (95\% \text{ saturation confidence interval})/(2\times 1.96) \tag{32}$$

Since the saturation volume V_s is also constant, sigma can easily be estimated by Equation (33) and the summary of the parameter fitting is shown in **Table 1**.

$$\sigma = Sd(\infty)/V_s \tag{33}$$

3.2. Gompertz Diffusion Fitting

The Gompertz model can be fitted by common statistic features, such as maximum verisimilitude (see Gutierrez et al., 2008), but the lack of equal time distribution of data make it difficult. So a quadratic fitting method was development using the SSPS software. By taking natural logarithm and arranging it we get Equation (34).

$$\ln E[V_t] = A - Bx - Cx^2 \tag{34}$$

$$\text{with} \quad A = \theta - \sigma^2/(4k),\ B = \theta - \sigma^2/(2k) - \ln(V_0),\ C = \sigma^2/(4k) \ \text{ and } \ x = e^{-kt} \tag{35}$$

Given a value for k, a quadratic fitting for e^{-kt} and e^{-2kt} was done estimating the value of A, B and C until a common value for θ was obtained from A and B, determining the estimation for θ, k and σ. The deterministic parameter only requires a linear fitting with e^{-kt}. Both fittings were done for the initial value V_0 = 32 (m^3/ha) and the results are summarized in **Table 2**.

3.3. Wood Price Diffusion Fitting Equations and Regeneration Costs

The stumpage stand price Brownian diffusion parameters are estimated from saw logs and pulp logs exportation prices (see **Appendix B**). The summary of Brown diffusion parameters for the stumpage price and the regeneration cost of radiata Pine Stands are given in **Table 3**, see Navarrete & Bustos (2013).

4. Stochastic Optimal Pine Harvestings Results

4.1. Logistic Wood Stock and Brown Stumpage Price models

The deterministic optimal solution given by $V^* = (\alpha + \mu - i)/(\gamma\mu)$, for the simple rotation model and by equation (36) for the multiple rotation model, see Navarrete (2013).

Table 1. Logistic fitting parameters.

Models	Drift Parameter μ	Saturation Volume	Saturation Parameter γ	Volatility Parameter σ
Stochastic	0.191	576.0	0.00174	0.12
Deterministic	0.191	576.0	0.00174	

Table 2. Gompertz diffusions parameters estimations.

Parameters	Saturation	Drift	Drift	Volatility	
Models	V_s	k	Θ	σ	R^2
Gompertz	653.3	0.102	6.538	0.151	0.992
Deterministic	653.3	0.102	6.538		

Table 3. Radiata pine price diffusion parameters and capital and regeneration stands costs.

Price stumpage drift	α	2.9%
Price stumpage volatility	β	15.9%
Actual stumpage log price	P_T	39.74 US$/ha
Initial stumpage price	P_0	21.43 US$/ha
Risky rate of Capital	WACC	12%
Stands regeneration cost	C	882 US$/ha
Stands cost per unit initial price	$c = C/P_0$	41.16

Source: Appendix B.

$$V^* = \frac{(\alpha+\mu-r_t)+\sqrt{(\alpha+\mu-r_t)^2+4\mu\gamma c r_t \mathrm{e}^{-\alpha t}}}{2\mu\gamma} \tag{36}$$

The solution of the Vicksell and Faustmann model under Brown price and Logistic wood stock diffusion requires the solution of the differential Equations (8) and (19) for $\mu(V)=\mu V(1-\gamma V)$ and $\sigma(V)=\sigma V$. The positive increasing function $\psi(V)$ for the Vicksell model, is the solution of the homogenous component (37) of the differential Equation (8) with $r_T = i$ in the Vicksell case.

The solution of Equation (37) is given by the Kummer's confluent hyper geometric function, expression (38) with the positive root θ by Equation (39), $\psi(V)$ also must fulfill the smooth pasting condition (12), which was programmed in Mapple 15, see Navarrete (2013).

$$\left(\frac{1}{2}\right)\sigma^2\overline{V^2}F''(\bar{V})+\left[\mu\bar{V}(1-\gamma\bar{V})+\beta\sigma\bar{V}\right]F'(\bar{V})-(r_T-\alpha)F(\bar{V})=0 \tag{37}$$

$$\Psi(\bar{V})=V^\theta KummerM\left\{\frac{2\mu\gamma\bar{V}}{\sigma^2},\theta,2\theta+\frac{2(\mu+\beta\sigma)}{\sigma^2}\right\} \tag{38}$$

$$\theta=\frac{1}{2}-\frac{\mu}{\sigma^2}-\frac{\beta}{\sigma}+\sqrt{\left(\frac{1}{2}-\frac{\mu}{\sigma^2}-\frac{\beta}{\sigma}\right)^2+\frac{2(r_T-\alpha)}{\sigma^2}} \tag{39}$$

The Faustmann stochastic optimum requires the solution of the differential Equation (16), so the positive function $\psi(V)$ is the solution of its homogenous Equation (37) for the parameter r_T and also fulfill the smooth pasting condition, (22). For a given value of the capitalized interest $r_T = i/(1-\mathrm{e}^{-iT})$ Equation (37) can be solved using the same methodology of the Vicksell solution, generating a family of optimal cuts for the different parameters r_T. The optimum solution is obtained intersecting the optimal parametric solution for different T with the corresponding expected value logistical modify diffusion, (14) for $\bar{V}$, see Figure 2. The summary of all optimal cuts results for the aggregate 30/35 site index series of the multiple and simple rotation harvest model under Brown price and Logistic wood stock diffusions is given in Table 4.

The simple optimal cut is a better explanation of the company cut policy, since it only underestimates it in a 2.5%. As expected the Faustmann optimal cut is lower and underestimate the company policy in 11.5%. The deterministic optimal cuts are even worse and underestimate the company policy in 23.4% in the simple model and in 25.6% in the multiple cases. Finally the Logistic wood stock underestimates the saturation Volume in 4.2%.

4.2. Gompertz Wood Stock and Brown Stumpage Price Case

The deterministic optimal solution, is given by $V^* = \mathrm{e}^{(\alpha+k\theta-i)/k}$ for the simple rotation model and for Equation (41) for the multiple rotation case the optimal is obtained intercepting condition (40) with Equation (27). See Navarrete (2013).

$$\ln(V)=\frac{(\alpha+k\theta-r_T)}{k}+\frac{r_T c\mathrm{e}^{-\alpha t}}{kV} \tag{40}$$

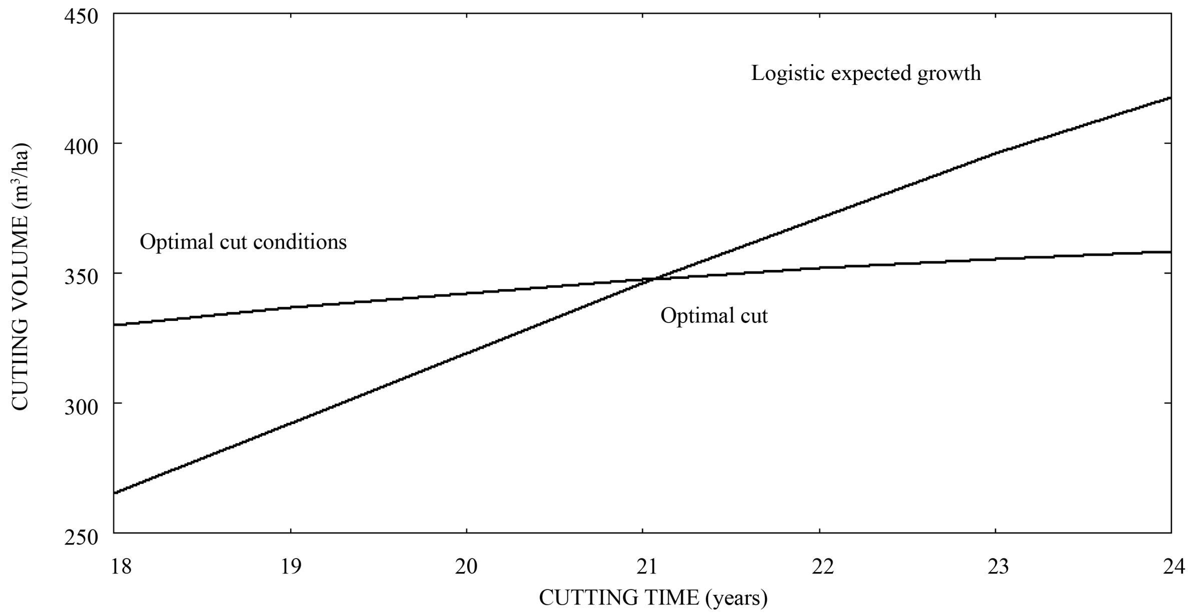

Figure 2. Faustmann logistic optimal cut.

Table 4. Optimal harvest rotation optimal results, Brown price logistic wood stock diffusion.

Optimum Policy	Simple m^3/ha	Harvest %	Multiple m^3/ha	Harvest %	Saturation m^3/ha	Volume %
Company	392.9	100.0	392.9	100.0	601.6	100
Deterministic	300.9	−23.4	292.3	−25.6		
Stochastic	383.1	−2.5	347.6	−11.5	576	−4.3

The solution of the Vicksell and Faustmann model under Brown price and Gompertz wood stock diffusion requires the solution of the differential Equations, (8) and (16) for $\mu(V)=kV(\theta-\ln V)$ and $\sigma(V)=\sigma V$. The stochastic increasing function $\psi(V)$, for these cases is given by the solution of the homogenous Equation (41) with $r_T = i$ for the simple rotation case and the capitalized rate of interest r_T for the Faustmann case.

$$\left(\frac{1}{2}\right)\sigma^2\bar{V}^2F''(\bar{V})+\left[k\bar{V}\left(\theta-\ln(\bar{V})\right)+\beta\sigma\bar{V}\right]F'(\bar{V})-(r_T-\alpha)F(\bar{V})=0 \tag{41}$$

Replacing the parameters given in Equations (42) in (41), gives the differential Equation (43) of the Exponential, Ornstein-Uhlembeck diffusion whose positive increasing solution $\psi(V)$ is given by Equation (44) (see Johnson, 2005), with: $a=(r-\alpha)/(2k)$, $b = 0.5$ and $z=(k/\sigma^2)\left[\theta-\sigma^2/(2k)+\beta\sigma/k-\ln(V)\right]^2$

$$\bar{\theta}=\theta-\frac{\sigma^2}{2k}+\frac{\beta\sigma}{k},\quad \bar{r}_T=r_T-\alpha \tag{42}$$

$$\left(\frac{1}{2}\right)\sigma^2\bar{V}^2F''(\bar{V})+\left[k\left(\bar{\theta}-\ln(\bar{V})\right)+\frac{1}{2}\sigma^2\right]\bar{V}F'(\bar{V})-\bar{r}_TF(\bar{V})=0 \tag{43}$$

$$\Psi(\bar{V})=\begin{cases}\left[\Gamma(a+1-b)/\Gamma(1-b)\right]KummerU(a,b,z) & \bar{V}\le e^{\theta}\\ KummerM(a,b,z) & \bar{V}\ge e^{\theta}\end{cases} \tag{44}$$

The optimal solutions also requires that $\psi(V)$ fulfills the smooth pasting condition (38), which is programmed in Mapple 15 generating one optimal cut solution for the Vicksell model and a family of optimal cut condition parametrized by T for the Faustmann case. In this late case the optimal solution is obtain intercepting the family with the expected volume of the Gompertz modify diffusion (20), see **Figure 3**.

The deterministic and stochastic optimum were programed in Maple 15, using in this case the *KummerU* function of the program, the results are summarized in **Table 5**. These results show that the simple stochastic rotation model under Brown Price and Gompertz wood stock diffusion is a better explanation of the company cut policy, since it only overestimates it in 1.2%. As expected, the Faustmann model has lower optimal cut and underestimate it in 9.2% and the deterministic optimal cut is even lower, and underestimate the company policy in 27.9% in the simple case and in 33.1% in the multiple cases. Finally, the Gompertz wood stock overestimated the Saturation volume in 8.6%.

5. Conclusions

1) The optimal cut company policy validates the use of the simple stochastic rotations model under Bown price and Logistic or Gompertz wood stock diffusion.
2) The discrepancy in the theoretical and practical cut policy can be explained by the preference that the business policy gives to shorter rotations periods 25 or less years due to the high organizational risk of the industrial sector in Chile.
3) The Gompertz and Logistic diffusion models present small estimation differences in the growing phase of wood stock, but significant differences in the saturation volume of the wood stock, which should be crucial in the model diffusion selection.

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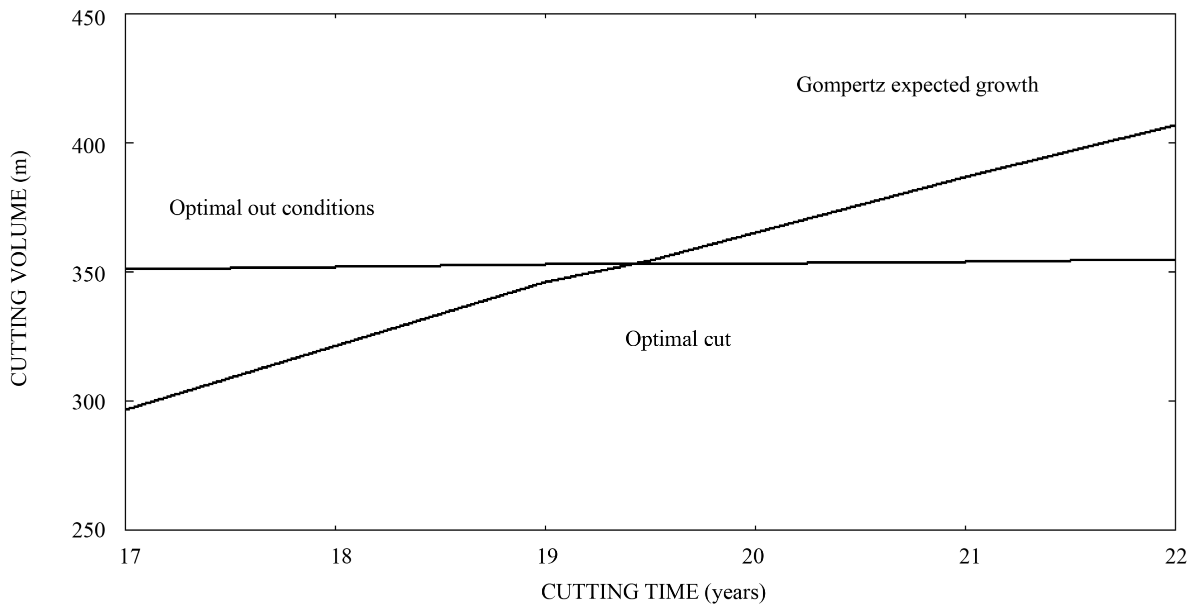

Figure 3. Faustmann gompertz optimal cut.

Table 5. Optimal harvest rotation optimal results, Brown price Gompertz wood stock diffusion.

Optimum Policy	Simple mts^3/ha	Rotation %	Multiple mts^3/ha	Rotation %	Saturation mts^3/ha	Volume%
Company	392.9	100.0	392.9	100.0	601.6	100
Deterministic	283.2	−27.9	262.8	−33.1		
Stochastic	397.6	1.2	353.2	−10.1	653.3	8.6

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Appendix A: Mininco Radiata Pine Stands Harvest Data

OT	FECHA_INV	EDAD	DENSIDAD	INDICE_SIT	POD	DEBO	INDUST	COMERC	PULPA	VOLTOT
Number	Date	Years	Number	mts	mts^3/ha	mts^3/ha	mts^3/ha	mts^3/ha	mts^3/ha	mts^3/ha
102809	25-01-2005	20,069	412	31,00	30,211	13,495	195,12	8,336	69,645	316,807
103160	08-02-2005	20,106	305	30,60	60,217	36,52	211,321	10,159	39,562	357,779
102805	22-02-2005	20,144	345	31,00	43,67	15,108	178,829	10,494	55,661	303,762
95972	29-08-2004	20,664	294	30,70	51,879	9,969	168,742	6,612	44,13	281,332
51601	09-09-1999	20,692	406	30,40	47,735	23,134	182,41	6,609	67,564	327,452
51635	12-09-1999	20,700	440	30,80	49,289	33,646	247,323	9,609	77,83	417,697
51691	18-09-1999	20,717	533	30,50	23,009	11,101	213,896	10,43	103,368	361,804
71818	15-10-2001	20,792	761	30,90	49,298	26,365	332,527	30,804	129,435	568,429
83880	15-01-2003	21,042	299	30,70	88,091	21,753	231,1	19,339	38,159	398,442
84094	15-01-2003	21,042	359	30,90	46,783	19,625	190,715	8,831	60,556	326,510
102834	25-01-2005	21,069	300	30,70	66,272	10,348	191,916	8,761	45,23	322,527
102744	25-01-2005	21,069	350	31,00	57,347	27,673	184,91	9,881	53,279	333,090
91528	30-01-2004	21,083	323	30,60	41,753	14,123	156,805	28,112	51,448	292,241
75790	01-02-2002	21,086	275	30,90	89,538	34,915	150,87	29,031	35,425	339,779
102865	04-02-2005	21,094	342	30,80	127,279	34,777	214,458	9,074	41,394	426,982
75786	13-02-2002	21,119	325	31,00	53,673	26,437	143,21	50,131	51,291	324,742
83931	16-02-2003	21,128	250	30,70	27,758	17,649	121,4	21,834	40,96	229,601
75787	16-02-2002	21,128	348	30,80	68,175	20,528	166,942	44,564	54,404	354,613
90973	23-02-2004	21,147	393	30,60	52,687	19,955	183,955	13,828	62,824	333,249
76333	06-03-2002	21,183	332	30,70	77,712	23,876	165,961	72,01	46,151	385,710
91641	06-03-2004	21,183	410	30,70	0,529	18,27	313,335	22,392	61,533	416,059
91504	08-03-2004	21,189	294	31,00	73,553	14,884	183,809	21,771	43,984	338,001
91501	17-03-2004	21,214	284	31,00	79,132	13,879	187,021	15,73	39,008	334,770
91151	20-03-2004	21,231	374	31,00	52,099	22,609	189,536	11,727	59,183	335,154
76319	24-03-2002	21,233	236	30,80	83,656	28,056	144,961	4,015	29,774	290,462
91161	26-03-2004	21,239	364	30,60	20,259	15,781	169,953	9,646	63,734	279,373
83406	23-01-2003	21,639	515	30,90	21,687	14,347	186,542	16,015	94,517	333,108
52198	16-11-1999	21,878	1029	29,90	2,217	1,009	371,784	11,957	239,828	626,795
52203	22-11-1999	21,894	609	30,30	42,256	9,958	231,214	10,5	124,584	418,512
63134	13-12-2000	21,953	342	30,10	52,596	41,286	252,125	6,077	49,345	401,429
74184	19-12-2001	21,969	279	30,60	81,005	22,015	187,372	5,492	39,47	335,354
66444	07-03-2001	22,186	313	30,90	89,457	9,718	208,89	6,233	49,866	364,164
83886	01-02-2003	23,086	346	30,90	35,788	46,389	274,181	19,238	51,227	426,823
83885	19-02-2003	23,136	304	30,80	100,84	8,561	227,416	18,014	43,074	397,905
71608	03-10-2001	23,758	800	30,90	0	0	338,759	42,563	146,526	527,848
74494	11-12-2001	23,947	213	30,90	77,12	15,116	204,076	7,144	28,522	331,978
101906	14-01-2005	24,039	265	30,70	126,486	7,274	253,543	22,04	34,144	443,487

Continued

113755	29-08-2005	24,664	672	30,60	93,303	33,622	348,677	15,913	118,159	609,674
87729	06-10-2003	24,767	446	30,80	4,102	1,139	446,01	20,833	74,681	546,765
101738	20-01-2005	20,056	344	31,70	97,009	21,861	226,91	25,476	46,081	417,337
123608	20-01-2006	20,056	447	31,70	1,289	0	301,841	31,847	73,527	408,504
101747	26-01-2005	20,072	310	31,80	62,756	12,844	163,095	18,867	47,627	305,189
101748	27-01-2005	20,075	317	31,50	67,198	13,985	168,31	28,119	56,473	334,085
123355	28-01-2006	20,078	890	31,70	0	0	340,058	43,398	154,206	537,662
101749	01-02-2005	20,086	295	31,30	54,651	14,83	172,718	10,396	43,363	295,958
102750	05-02-2005	20,097	339	31,40	83,138	29,424	191,109	7,36	45,999	357,030
123632	06-02-2006	20,100	357	31,90	142,767	29,785	237,26	9,796	46,357	465,965
123226	09-02-2006	20,108	502	31,10	41,176	44,536	276,147	19,468	74,52	455,847
123635	23-02-2006	20,147	300	31,20	114,971	20,195	193,637	10,067	39,382	378,252
103466	24-02-2005	20,150	339	32,00	40,519	37,1	217,016	21,458	51,34	367,433
48607	04-03-1999	20,178	397	31,20	68,764	12,871	227,406	6,839	63,118	378,998
48597	18-03-1999	20,217	448	31,40	85,781	18,874	250,142	6,616	68,97	430,383
48606	23-03-1999	20,231	547	31,50	74,666	14,134	270,051	8,732	93,449	461,032
48611	05-04-1999	20,264	395	31,90	77,063	14,042	235,185	8,001	68,931	403,222
48627	06-04-1999	20,267	420	31,40	57,08	15,927	220,74	6,045	75,991	375,783
48636	15-04-1999	20,292	326	31,40	95,803	10,146	215,994	6,66	69,367	397,970
95971	27-08-2004	20,658	328	31,90	55,705	11,822	200,075	8,366	51,535	327,503
98360	26-09-2004	20,739	351	31,90	91,606	10,282	235,839	34,439	51,152	423,318
47911	22-12-1998	20,978	785	31,40	32,303	16,294	258,41	19,006	194,812	520,825
47912	23-12-1998	20,981	629	31,40	38,985	11,536	237,974	14,113	152,54	455,148
101575	13-01-2005	21,036	404	32,00	92,721	14,187	255,165	16,403	62,515	440,991
102567	17-01-2005	21,047	417	31,10	103,542	18,677	241,928	16,069	60	440,216
101744	19-01-2005	21,053	406	31,80	88,98	11,174	255,961	12,042	64,296	432,453
102563	20-01-2005	21,056	469	31,30	88,279	31,801	258,86	21,657	69,483	470,080
102747	21-01-2005	21,058	360	31,30	46,999	26,463	194,858	8,788	57,079	334,187
101573	21-01-2005	21,058	333	32,00	55,895	13,776	192,788	17,102	53,577	333,138
101745	24-01-2005	21,067	372	31,70	80,019	14,1	245,135	10,045	57,013	406,312
101574	25-01-2005	21,069	333	31,30	70,24	20,794	203,195	24,886	56,044	375,159
101833	27-01-2005	21,075	317	31,20	110,331	6,772	196,835	47,767	44,234	405,939
91460	27-01-2004	21,075	279	31,30	86,05	33,086	188,059	18,132	37,383	362,710
91467	27-01-2004	21,075	358	31,30	42,62	18,72	199,167	19,963	57,223	337,693
91529	29-01-2004	21,081	315	31,10	53,811	10,9	161,087	31,303	50,264	307,365
91465	29-01-2004	21,081	274	31,20	82,99	10,868	192,224	8,701	39,322	334,105
75796	31-01-2002	21,086	317	31,80	75,935	28,873	170,227	8,407	49,172	332,614
102824	02-02-2005	21,089	290	31,80	64,403	9,414	172,096	26,284	47,789	319,986
75887	09-02-2002	21,108	368	31,30	52,133	21,99	179,072	7,763	61,943	322,901
103461	09-02-2005	21,108	297	31,90	82,518	15,045	203,406	30,78	40,854	372,603

Continued

75889	10-02-2002	21,111	348	31,40	63,778	27,662	180,999	6,773	56,997	336,209
90970	17-02-2004	21,131	428	32,00	56,244	16,199	198,662	18,414	68,095	357,614
102833	22-02-2005	21,144	373	31,30	71,751	19,283	229,216	8,982	56,629	385,861
102832	23-02-2005	21,147	377	31,80	64,873	16,667	230,532	9,745	60,034	381,851
91308	15-03-2004	21,208	333	31,40	7,169	35,563	195,655	108,045	50,411	396,843
91152	15-03-2004	21,208	290	32,00	96,706	33,638	184,155	9,495	39,262	363,256
91508	17-03-2004	21,214	277	31,90	83,842	21,53	175,696	10,507	40,067	331,642
91138	19-03-2004	21,219	352	31,10	43,032	24,143	213,379	11,063	51,024	342,641
91499	19-03-2004	21,219	354	31,70	60,3	23,848	193,449	12,761	56,861	347,219
76314	21-03-2002	21,225	295	31,10	83,129	29,704	168,314	6,804	39,028	326,979
91150	30-03-2004	21,250	335	31,10	54,334	28,177	137,598	74,463	49,396	343,968
52202	20-11-1999	21,889	1000	31,30	0	0	375,916	11,432	192,103	579,451
52204	26-11-1999	21,906	915	31,20	0	0	271,492	10,512	213,957	495,961
56179	14-03-2000	22,206	546	31,70	89,278	20,81	305,403	8,982	99,265	523,738
83523	17-01-2003	23,047	211	31,40	77,772	4,129	153,92	32,647	31,203	299,671
75661	05-02-2002	23,097	298	31,10	153,673	8,138	284,036	53,474	36,826	536,147
76843	12-03-2002	23,200	550	31,90	60,901	22,524	322,74	40,764	98,317	545,246
96629	26-08-2004	24,656	222	31,60	109,406	14,078	239,825	13,152	26,738	403,199
96628	28-08-2004	24,661	221	32,00	110,83	10,141	242,607	10,845	27,866	402,289
96630	30-08-2004	24,667	225	31,60	92,591	10,419	195,494	27,384	30,712	356,600
81275	25-09-2002	24,736	447	31,30	26,623	52,339	418,624	23,667	68,454	589,707
114503	15-09-2005	25,708	347	31,50	85,776	30,121	316,837	13,993	48,589	495,316
101751	11-01-2005	20,031	313	32,10	100,369	9,866	200,563	22,615	44,89	378,303
101746	19-01-2005	20,053	303	32,40	55,155	11,971	171,863	27,424	47,264	313,677
101753	20-01-2005	20,061	290	32,10	100,957	10,857	197,157	7,7	40,732	357,403
103162	25-01-2005	20,069	363	34,50	0	0	308,817	43,479	62,114	414,410
102765	02-02-2005	20,089	359	33,10	43,378	20,648	215,688	13,047	60,672	353,433
103465	23-02-2005	20,147	324	32,70	35,602	33,756	213,888	43,265	49,633	376,144
127213	25-02-2006	20,153	312	32,30	61,569	15,948	125,533	73,06	45,674	321,784
124619	25-02-2006	20,153	414	35,10	0	0	304,078	23,043	93,149	420,270
48637	13-04-1999	20,286	438	33,70	109,062	17,36	298,884	8,893	80,999	515,198
48633	16-04-1999	20,294	386	32,10	75,26	16,378	304,09	6,758	61,994	464,480
95970	30-08-2004	20,667	299	32,90	62,127	14,75	211,774	8,368	44,546	341,565
98361	28-09-2004	20,745	383	32,50	78,934	9,75	262,531	30,704	57,672	439,591
101740	07-01-2005	21,019	435	32,20	90,006	12,714	267,781	22,698	72,831	466,030
83405	16-01-2003	21,045	339	32,20	46,08	22,387	180,907	13,669	58,606	321,649
102843	18-01-2005	21,050	296	33,00	110,036	13,61	258,581	10,547	39,962	432,736
102795	19-01-2005	21,053	454	32,20	33,334	10,63	235,047	9,416	82,856	371,283
75754	08-02-2002	21,106	421	34,20	93,435	9,237	280,858	21,142	69,074	473,746
84120	15-02-2003	21,125	259	34,80	80,143	12,525	181,992	66,744	38,426	379,830

Continued

54835	26-02-2000	21,156	335	32,90	72,231	12,019	217,648	7,542	72,837	382,277
112964	04-08-2005	21,594	552	33,20	51,201	18,57	253,839	52,399	97,698	473,707
62236	27-09-2000	21,742	400	32,10	122,383	20,963	310,676	10,81	76,532	541,364
81497	30-09-2002	21,750	299	32,90	0,999	50,428	316,128	14,549	40,111	422,215
83616	21-02-2003	22,142	292	32,50	87,783	26,583	194,151	29,612	41,126	379,255
74324	15-12-2001	22,958	347	32,70	71,151	28,513	177,427	49,93	61,445	388,466
75955	22-02-2002	23,144	236	32,10	103,693	38,456	183,181	5,162	30,643	361,135
103438	24-02-2005	24,150	260	35,00	27,641	59,521	352,155	18,261	35,659	493,237
76837	18-03-2002	24,217	515	32,20	16,069	8,318	468,02	8,436	79,23	580,073
124604	28-02-2006	25,161	347	32,10	111,71	41,25	255,813	18,786	52,947	480,506

Notes: TVOL = Total Volume (mts3/ha); TVOL = POD + DEBO + INDUST + COMERC + PULPA POD + DEBO = Wood thinning cuts; Logs composition: Saw logs = INDUST + COMERC = 83.9%; Pulp logs = PULP + POD + DEBO = 16.1%; Saturation volume Vs = 601.6 m^3/ha, (3% stands biggest TVOL average).

Appendix B: Radiata Pine Log Prices

Table B1. Nominal Radiata Pine log exportation prices.

YEARS	Saw logs	Pulp logs	YEARS	Saw logs	Pulp logs
	US$/$mts^3$	US$/$mts^3$		US$/$mts^3$	US$/$mts^3$
1985	32.0	27.0	1997	62.0	55.0
1986	34.0	28.0	1998	52.0	54.0
1987	39.0	27.0	1999	49.0	53.0
1988	45.0	27.0	2000	46.0	42.0
1989	43.0	27.0	2001	48.0	34.0
1990	49.0	32.0	2002	46.0	41.6
1991	51.0	40.0	2003	45.9	37.4
1992	47.0	40.0	2004	48.6	33.0
1993	85.0	38.0	2005	57.0	33.5
1994	63.0	46.0	2006	60.0	36.0
1995	67.0	43.0	2007	63.0	40.0
1996	65.0	52.0			

Source CONAF-INFOR Chile.

The Price diffusion parameters can very easily be calculated by making the following logarithmic transformation $p_t = \ln\left(p_t/p_{t-1}\right)$. The parameters are given by **Table B2**.

$$\alpha = \bar{p} = \sum \frac{p_t}{n} = \left(\alpha - \sigma^2/2\right),\ \ \beta^2 = \sum \frac{\left(p_t - \bar{p}\right)^2}{n-1}$$

Table B2. Log summary.

Summary	Stumpage logs	Saw logs	Pulp logs
Percentage	100%	83.9	16.1
Price drift α	2.9%	3.08	1.79
Price volatility β	15.9%	16.52	12.74
Average Price	39.74		