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New Extension of Unified Family of Apostol-Type Polynomials and Numbers

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Abstract

The purpose of this paper is to introduce and investigate new unification of unified family of Apostol-type polynomials and numbers based on results given in [1] [2]. Also, we derive some properties for these polynomials and obtain some relationships between the Jacobi polynomials, Laguerre polynomials, Hermite polynomials, Stirling numbers and some other types of generalized polynomials.

Keywords

Euler, Bernoulli and Genocchi Polynomials, Stirling Numbers, Laguerre Polynomials, Hermite Polynomials

1. Introduction

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{C}$ and the generalized Euler polynomials are defined by (see [3]):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!}, \left(|t| < 2\pi; 1^{\alpha} := 1\right) \tag{1.1}$$

and

$$\left(\frac{t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!}, \left(|t| < \pi; 1^{\alpha} := 1\right), \tag{1.2}$$

where $\mathbb{C}$ denotes the set of complex numbers.

Recently, Luo and Srivastava [4] introduced the generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ and the generalized Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ as follows.

Definition 1.1. (*Luo and Srivastava* [4]) *The generalized Apostol-Bernoulli polynomials* $B_n^{(\alpha)}(x;\lambda)$ *of order* $\alpha \in \mathbb{C}$ *are defined by the generating function*

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$$

$$\left(|t| < 2\pi \text{ when } \lambda = 1; |t| < |\log\lambda|, \text{when } \lambda \neq 1; 1^{\alpha} := 1\right). \tag{1.3}$$

Definition 1.2. (*Luo* [5]) *The generalized Apostol-Euler polynomials* $E_n^{(\alpha)}(x;\lambda)$ *of order* $\alpha \in \mathbb{C}$ *are defined by the generating function*

$$\left(\frac{t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$$

$$\left(|t| < \pi \text{ when } \lambda = 1; |t| < |\log(-\lambda)|, \text{when } \lambda \neq 1; 1^{\alpha} := 1\right). \tag{1.4}$$

Natalini and Bernardini [6] defined the new generalization of Bernoulli polynomials in the following definition.

Definition 1.3. *The generalized Bernoulli polynomials* $B_n^{[m-1]}(x)$, $m \in \mathbb{N}$, *are defined, in a suitable neighbourhood of* $t = 0$ *by means of generating function*

$$\frac{t^m e^{xt}}{e^t - \sum_{l=0}^{m-1}\frac{t^l}{l!}} = \sum_{n=0}^{\infty} B_n^{[m-1]}(x)\frac{t^n}{n!}. \tag{1.5}$$

Recently, Tremblay *et al*. [7] investigated a new class of generalized Apostol-Bernoulli polynomial as follows.

Definition 1.4. *The generalized Apostol-Bernoulli polynomials* $B_n^{[m-1,\alpha]}(x;\lambda)$ *of order* $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$, *are defined, in a suitable neighbourhood of* $t = 0$ *by means of generating function*

$$\left(\frac{t^m}{\lambda e^t - \sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}. \tag{1.6}$$

Also, Sirvastava *et al*. [1] introduced a new interesting class of Apostol-Bernoulli polynomials that are closely related to the new class that we present in this paper. They investigated the following form.

Definition 1.5. *Let* $a,b,c \in \mathbb{R}^+ (a \neq b)$ *and* $n \in \mathbb{N}_0$. *Then the generalized Bernoulli polynomials* $\mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ *are defined by the following generating function*:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}$$

$$\left(\left|t\log\left(\frac{a}{b}\right)\right| < |\log\lambda|; 1^{\alpha} := 1\right). \tag{1.7}$$

This sequel to the work by Sirvastava *et al*. [2] introduced and investigated a similar generalization of the family of Euler polynomials defined as follows.

Definition 1.6. *Let* $a,b,c \in \mathbb{R}^+ (a \neq b)$ *and* $n \in \mathbb{N}_0$. *Then the generalized Euler polynomials* $\mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)$ *of order* $\alpha \in \mathbb{C}$ *are defined by the following generating function*

$$\left(\frac{t}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}$$

$$\left(\left|t\log\left(\frac{a}{b}\right)\right| < |\log(-\lambda)|; 1^{\alpha} := 1\right). \tag{1.8}$$

It is easy to see that setting $a=1$ and $b=c=e$ in (1.8) would lead to Apostol-Euler polynomials defined by (1.4). The case where $a=1$ has been studied by Luo *et al.* [8].

In Section 2, we introduce the new extension of unified family of Apostol-type polynomials and numbers that are defined in [9]. Also, we determine relations between some results given in [1] [3] [7] [10] [11] and our results. Moreover, we introduce some new identities for polynomials defined in [9]. In Section 3, we give some basic properties of the new unification of Apostol-type polynomials and numbers. Finally in Section 4, we introduce some relationships between the new unification of Apostol-type polynomials and other known polynomials.

2. Unification of Multiparameter Apostol-Type Polynomials and Numbers

Definition 2.1. *Let* $a,b,c\in\mathbb{R}^{+}\left(a\neq b\right)$, $n\in\mathbb{N}_0$ *and* $m\in\mathbb{N}$. *Then the new unification of Apostol-type polynomials* $M_n^{[m-1,r]}\left(x;k;a,b,c;\bar{\alpha}_r\right)$ *are defined, in a suitable neighbourhood of* $t=0$ *by means of generating function*

$$F_{\bar{\alpha}_r}^{[m-1,r]}=\frac{t^{rkm}2^{rm(1-k)}c^{xt}}{\prod_{i=0}^{r-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}=\sum_{n=0}^{\infty}M_n^{[m-1,r]}\left(x;k;a,b,c;\bar{\alpha}_r\right)\frac{t^n}{n!}$$

$$\left(\left|t\log\left(\frac{b}{a}\right)\right|<2\pi \text{ when } m=1 \text{ and } \alpha_i=1;\left|t\log\left(\frac{b}{a}\right)\right|<\left|\log\left(\alpha_i\right)\right| \text{ when } m=1 \text{ and } \alpha_i\neq 1;\forall\, i=0,1,\cdots,r-1\right),\quad (2.1)$$

where $k\in N_0;\ r\in\mathbb{C};\ \bar{\alpha}_r=\left(\alpha_0,\alpha_1,\cdots,\alpha_{r-1}\right)$ *is a sequence of complex numbers.*

Remark 2.1. *If we set* $x=0$ *in* (2.1), *then we obtain the new unification of multiparameter Apostol-type numbers, as*

$$M_n^{[m-1,r]}\left(0;k;a,b,c;\bar{\alpha}_r\right)=M_n^{[m-1,r]}\left(k;a,b,c;\bar{\alpha}_r\right). \tag{2.2}$$

The generating function in (2.1) gives many types of polynomials as special cases, for example, see **Table 1**.

Remark 2.2. *From NO.* 13 *in* **Table 1** *and* ([9], **Table 1**), *we can obtain the polynomials and the numbers given in* [12]-[16].

3. Some Basic Properties for the Polynomial $M_n^{[m-1,r]}\left(x;k;a,b,c;\bar{\alpha}_r\right)$

Theorem 3.1. *Let* $a,b,c\in\mathbb{R}^{+}\left(a\neq b\right)$ *and* $x\in R$. *Then*

$$M_n^{[m-1,r]}\left(x+y;k;a,b,c;\bar{\alpha}_r\right)=\sum_{l=0}^{n}\binom{n}{l}x^{n-l}\left(\ln c\right)^{n-l}M_l^{[m-1,r]}\left(y;k;a,b,c;\bar{\alpha}_r\right). \tag{3.1}$$

$$M_n^{[m-1,r]}\left(x+r;k;a,b,c;\bar{\alpha}_r\right)=M_n^{[m-1,r]}\left(x;k;\frac{a}{c},\frac{b}{c},c;\bar{\alpha}_r\right). \tag{3.2}$$

Proof. For the first equation, from (2.1)

$$\sum_{n=0}^{\infty}M_n^{[m-1,r]}\left(x+y;k;a,b,c;\bar{\alpha}_r\right)\frac{t^n}{n!}=\frac{t^{rkm}2^{rm(1-k)}c^{xt}}{\prod_{i=0}^{r-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}c^{yt}=\sum_{j=0}^{\infty}\frac{\left(ty\ln c\right)^j}{j!}\sum_{l=0}^{\infty}M_l^{[m-1,r]}\left(x;k;a,b,c;\bar{\alpha}_r\right)\frac{t^l}{l!},$$

using Cauchy product rule, we can easily obtain (3.1).

For the second Equation (3.2), from (2.1)

$$\sum_{n=0}^{\infty}M_n^{[m-1,r]}\left(x+r;k;a,b,c;\bar{\alpha}_r\right)\frac{t^n}{n!}=\frac{t^{rkm}2^{rm(1-k)}}{\prod_{i=0}^{r-1}\left(\alpha_i\left(\frac{b}{c}\right)^t-\left(\frac{a}{c}\right)^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}c^{xt}=\sum_{n=0}^{\infty}M_n^{[m-1,r]}\left(x;k;\frac{a}{c},\frac{b}{c},c;\bar{\alpha}_r\right)\frac{t^n}{n!}.$$

Table 1. Special cases.

1	setting $k=1, \alpha_i=\lambda, \quad i=0,1,\cdots,r-1$, hence if $m=1$ in (2.1)	$M_n^{[0,r]}(x;1;a,b,c;\lambda)=\mathfrak{B}_n^{(r)}(x;\lambda;a,b,c)$ (generalized Bernoulli polynomials of order *r*, see [2])
2	setting $k=0, \alpha_i=-\lambda, \quad i=0,1,\cdots,r-1$, hence if $m=1$ in (2.1)	$M_n^{[0,r]}(x;0;a,b,c;-\lambda)=(-1)^r \mathfrak{E}_n^{(r)}(x;\lambda;a,b,c)$ (generalized Euler polynomials of order *r*, see [2])
3	setting $\alpha_i=\beta, \quad i=0,1,\cdots,r-1, c=b$, hence if $m=1$ in (2.1)	$M_n^{[0,r]}(x;k;a,b,b;\beta)=y_{n,\beta}^{(r)}(x;k;a,b)$ (unification of Apostol-type polynomials of order *r*, see [12])
4	setting $k=1, t=t\ln a, x=\dfrac{x}{\ln a}, \alpha_i=\lambda$, $i=0,1,\cdots,r-1$, hence if $a=1, b=c^{\frac{1}{\ln a}}$ in (2.1)	$M_n^{[m-1,r]}\left(\dfrac{x}{\ln a};1,1,c^{\frac{1}{\ln a}},c;\lambda\right)=(\ln a)^{mr} B_n^{[m-1,r]}(x;c,a;\lambda)$ (generalized Bernoulli polynomials of order *r*, see [11])
5	setting $k=0, t=t\ln a, x=\dfrac{x}{\ln a}, \alpha_i=-\lambda$, $i=0,1,\cdots,r-1$, hence if $a=1, b=c^{\frac{1}{\ln a}}$ in (2.1)	$M_n^{[m-1,r]}\left(\dfrac{x}{\ln a};0;1,c^{\frac{1}{\ln a}},c;-\lambda\right)=(-1)^r(\ln a)^{mr} E_n^{[m-1,r]}(x;c,a;\lambda)$ (generalized Euler polynomials of order *r*, see [11])
6	setting $k=1, \alpha_i=1, \quad i=0,1,\cdots,r-1, a=1, b=e, c=e$, hence if $r=1$ in (2.1)	$M_n^{[m-1,1]]}(x;1;1,e,e,1)=B_n^{[m-1]}(x)$ (generalized Bernoulli polynomials, see [6])
7	setting $k=0, \alpha_i=-1, \quad i=0,1,\cdots,r-1, a=1, b=e, c=e$, hence if $r=1$ in (2.1)	$M_n^{[m-1,1]}(x;0;1,e,e;-1)=-E_n^{[m-1]}(x)$ (generalized Euler polynomials, see [6])
8	setting $k=1, \alpha_i=1$, $i=0,1,\cdots,r-1, a=1, b=e, c=e$ in (2.1)	$M_n^{[m-1,r]}(x;1;1,e,e;1)=B_n^{[m-1,r]}(x)$ (generalized Bernoulli polynomials of order *r*, see [10])
9	setting $k=0, \alpha_i=-1$, $i=0,1,\cdots,r-1, a=1, b=e, c=e$ in (2.1)	$M_n^{[m-1,r]}(x;0;1,e,e;-1)=(-1)^r E_n^{[m-1,r]}(x)$ (generalized Euler polynomials of order *r*, see [10])
10	setting $k=1, \alpha_i=-1$, $i=0,1,\cdots,r-1, a=1, b=e, c=e$ in (2.1)	$M_n^{[m-1,r]}(x;1;1,e,e;-1)=(-1)^r\left(\dfrac{1}{2}\right)^{rm} G_n^{[m-1,r]}(x)$ (generalized Genocchi polynomials of order *r*, see [10])
11	setting $k=1, \alpha_i=\lambda$, $i=0,1,\cdots,r-1, a=1, b=e, c=e$ in (2.1)	$M_n^{[m-1,r]}(x;1;1,e,e;\lambda)=B_n^{[m-1,r]}(x;\lambda)$ (generalized Apostol-Bernoulli polynomials of order *r*, see [7])
12	setting $k=0, \alpha_i=-\lambda$, $i=0,1,\cdots,r-1, a=1, b=e, c=e$ in (2.1)	$M_n^{[m-1,r]}(x;0;1,e,e;-\lambda)=(-1)^r E_n^{[m-1,r]}(x;\lambda)$ (generalized Apostol-Euler polynomials of order *r*, see [7])
13	setting $m=1, a=1, b=e, c=e$ in (2.1)	$M_n^{[0,r]}(x;k;1,e,e;-\bar{\alpha}_r)=M_n^{(r)}(x;k;\bar{\alpha}_r)$ (a new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, see [9])

Equating the coefficient of $\dfrac{t^n}{n!}$ on both sides, yields (3.2). □

Corollary 3.1. *If* $y=0$ *in* (3.1), *we have*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{\ell=0}^{n}\binom{n}{\ell}x^{n-\ell}(\ln c)^{n-\ell} M_\ell^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r) \tag{3.3}$$

$$=\sum_{\ell=0}^{n}\binom{n}{n-\ell}x^{\ell}(\ln c)^{\ell} M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r). \tag{3.4}$$

Theorem 3.2. *The following identity holds true, when* $m=1$ *and* $\alpha_i \neq 0$ *in* (2.1) $\forall i=0,1,\cdots,r-1$

$$M_n^{[0,r]}\left(r-x;k;a,b,c;\bar{\alpha}_r\right)=\frac{(-1)^{r(1-k)+n}}{\prod_{i=0}^{r-1}\alpha_i}\sum_{m=0}^{n}\binom{n}{m}\left(r\ln\left(\frac{ab}{c}\right)\right)^{n-m}M_m^{[0,r]}\left(x;k;a,b,c;\frac{1}{\bar{\alpha}_r}\right). \tag{3.5}$$

Proof. From (2.1)

$$\begin{aligned}
&\sum_{n=0}^{\infty}M_n^{[0,r]}\left(r-x;k;a,b,c;\bar{\alpha}_r\right)\frac{t^n}{n!}\\
&=\frac{t^{rk}2^{r(1-k)}c^{(r-x)t}}{\prod_{i=0}^{r-1}\left(\alpha_i b^t-a^t\right)}=\frac{(-1)^{r(1-k)}}{\left(b^t a^t\right)^r\prod_{j=0}^{r-1}\alpha_j}\frac{(-t)^{rk}2^{r(1-k)}c^{-xt}}{\prod_{i=0}^{r-1}\left(\frac{b^{-t}}{\alpha_i}-a^{-t}\right)}c^{rt}\\
&=\frac{(-1)^{r(1-k)}}{\prod_{j=0}^{r-1}\alpha_j}\left(\frac{ba}{c}\right)^{-rt}\sum_{m=0}^{\infty}M_m^{[0,r]}\left(x;k;a,b,c;\frac{1}{\bar{\alpha}_r}\right)\frac{(-t)^m}{m!}\\
&=\frac{(-1)^{r(1-k)}}{\prod_{j=0}^{r-1}\alpha_j}\sum_{\ell=0}^{\infty}\frac{\left(r\ln\left(\frac{ab}{c}\right)\right)^{\ell}}{\ell!}(-t)^{\ell}\sum_{m=0}^{\infty}M_m^{[0,r]}\left(x;k;a,b,c;\frac{1}{\bar{\alpha}_r}\right)\frac{(-t)^m}{m!}.
\end{aligned}$$

Hence, we can easily obtain (3.5). □

Remark 3.1. *If we put* $\alpha_i=\beta, i=0,1,\cdots,r-1$, $c=b$ *and* $r=\upsilon$ *in* (3.5), *then it gives* [[12], *Equation* (34)],

$$M_n^{[0,\upsilon]}\left(\upsilon-x;k;a,b,b;\beta\right)=\frac{(-1)^{\upsilon(1-k)+n}}{(\beta)^{\upsilon}}\sum_{m=0}^{n}\binom{n}{m}\left(\upsilon\ln a\right)^{n-m}M_m^{[0,\upsilon]}\left(x;k;a,b,b;\beta^{-1}\right),$$

where $M_m^{[0,\upsilon]}\left(x;k;a,b,b;\beta^{-1}\right)$ is the unification of the Apostol-type polynomials.

Theorem 3.3. *The unification of Apostol-type numbers satisfy*

$$M_n^{[m-1,r]}\left(k;a,b,c;\bar{\alpha}_r\right)=\sum_{l=0}^{n}\binom{n}{l}M_l^{[m-1,\ell]}(k;a,b,c;\bar{\alpha}_\ell)M_{n-l}^{[m-1,r-\ell]}\left(k;a,b,c;\bar{\alpha}_{r-\ell}\right). \tag{3.6}$$

Proof. When $x=0$ in (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}M_n^{[m-1,r]}\left(k;a,b,c;\bar{\alpha}_r\right)\frac{t^n}{n!}\\
&=\frac{t^{rkm}2^{rm(1-k)}}{\prod_{i=0}^{r-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}=\frac{t^{\ell km}2^{\ell m(1-k)}}{\prod_{i=0}^{\ell-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}\frac{t^{(r-\ell)km}2^{(r-\ell)m(1-k)}}{\prod_{i=\ell}^{r-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}\\
&=\sum_{\ell_1=0}^{\infty}M_{\ell_1}^{[m-1,\ell]}\left(k;a,b,c;\bar{\alpha}_\ell\right)\frac{t^{\ell_1}}{\ell_1!}\sum_{\ell_2=0}^{\infty}M_{\ell_2}^{[m-1,r-\ell]}\left(k;a,b,c;\bar{\alpha}_{r-\ell}\right)\frac{t^{\ell_2}}{\ell_2!}.
\end{aligned}$$

Using Cauchy product rule, we obtain (3.6). □

Theorem 3.4. *The following relationship holds true*

$$\sum_{k_1+k_2+\cdots+k_\ell=n}\prod_{i=1}^{\ell}\frac{M_{k_i}^{[m-1,r_i]}\left(x_i;k;a,b,c;\bar{\alpha}_{r_i}\right)}{k_1!k_2!\cdots k_\ell!}=\frac{1}{n!}M_n^{[m-1,|r|]}\left(|\mathbf{x}|;k;a,b,c;\bar{\alpha}_{|r|}\right), \tag{3.7}$$

where $|\mathbf{r}|=r_1+r_2+\cdots+r_\ell$ *and* $|\mathbf{x}|=x_1+x_2+\cdots+x_\ell$ *and* $\bar{\alpha}_{r_i}=\left(\alpha_{\sum_{j=1}^{i-1}r_j},\alpha_{\sum_{j=1}^{i-1}r_j+1},\cdots,\alpha_{\sum_{j=1}^{i}r_j-1}\right)$, $i=\{1,2,\cdots,\ell\}$.

Proof. Starting with (2.1), we get

$$\sum_{n=0}^{\infty}\left(M_n^{[m-1,|r|]}\left(|x|;k;a,b,c;\bar{\alpha}_{|r|}\right)\right)\frac{t^n}{n!}$$
$$=\frac{t^{|r|km}2^{|r|m(1-k)}c^{|x|t}}{\prod_{i=0}^{|r|-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}$$
$$=\frac{t^{r_1 km}2^{r_1 m(1-k)}c^{x_1 t}}{\prod_{i=0}^{r_1-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}\frac{t^{r_2 km}2^{r_2 m(1-k)}c^{x_2 t}}{\prod_{i=r_1}^{r_1+r_2-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}\cdots\frac{t^{r_\ell km}2^{r_\ell m(1-k)}c^{x_\ell t}}{\prod_{i=r_1+r_2+\cdots+r_{\ell-1}}^{r_1+r_2+\cdots+r_\ell-1}\left(\alpha_i b^t-a^t\sum_{\ell=0}^{m-1}\frac{t^\ell}{\ell!}\right)}$$
$$=\sum_{k_1=0}^{\infty}M_{k_1}^{[m-1,r_1]}\left(x_1;k;a,b,c;\bar{\alpha}_{r_1}\right)\frac{t^{k_1}}{k_1!}\sum_{k_2=0}^{\infty}M_{k_2}^{[m-1,r_2]}\left(x_2;k;a,b,c;\bar{\alpha}_{r_2}\right)\frac{t^{k_2}}{k_2!}\cdots\sum_{k_\ell=0}^{\infty}M_{k_1}^{[m-1,r_\ell]}\left(x_\ell;k;a,b,c;\bar{\alpha}_{r_\ell}\right)\frac{t^{k_\ell}}{k_\ell!}$$

Using Cauchy product rule on the right hand side of the last equation and equating the coefficients of t^n on both sides, yields (3.7). □

Using No. 13 in **Table 1**, we obtain *Nörlund*'s results, see [17] and *Carlitz*'s generalizations, see [18] by our approach in Theorem 3.5 and Theorem 3.6 as follows

Theorem 3.5. *For* $(\bar{\alpha}_r)^n=\left(\alpha_0^n,\alpha_1^n,\cdots,\alpha_{r-1}^n\right)$, *we have*

$$\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\right)^{s_i}M_\ell^{[0,r]}\left(x+\frac{\sum_{i=1}^{r}s_i}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)=n^{rk-\ell}M_\ell^{[0,r]}\left(nx+;k;1,e,e;\bar{\alpha}_r\right). \tag{3.8}$$

$$\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\right)^{s_i}M_{r+\ell}^{[0,r]}\left(x+\frac{\sum_{i=0}^{r}s_i}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)=n^{r(k-1)-\ell}\frac{(\ell+r)!}{\ell!}M_\ell^{[0,r]}\left(nx+;k-1;1,e,e;\bar{\alpha}_r\right). \tag{3.9}$$

Proof. For the first equation and starting with (2.1), we get

$$\sum_{\ell=0}^{\infty}\frac{(nt)^\ell}{\ell!}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\right)^{s_i}M_\ell^{[0,r]}\left(x+\frac{\sum_{i=1}^{r}s_i}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)$$
$$=\frac{(nt)^{rk}2^{r(1-k)}\mathrm{e}^{nxt}}{\prod_{i=0}^{r-1}\alpha_i^n\mathrm{e}^{nt}-1}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\mathrm{e}^t\right)^{s_i}=\frac{(nt)^{rk}2^{r(1-k)}\mathrm{e}^{(nx)t}}{\prod_{i=0}^{r-1}\alpha_i e^t-1}=n^{rk}\sum_{\ell=0}^{\infty}M_\ell^{[0,r]}\left(nx;k;1,e,e;\bar{\alpha}_r\right)\frac{t^\ell}{\ell!}.$$

Equating the coefficients of t^ℓ on both sides, yields (3.8).

For the second equation and starting with (2.1), we get

$$\sum_{\ell=0}^{\infty}\frac{(nt)^\ell}{\ell!}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\right)^{s_i}M_\ell^{[0,r]}\left(x+\frac{\sum_{i=1}^{r}s_i}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)$$
$$=\frac{n^{rk}t^r2^{-r}(t)^{r(k-1)}2^{(2-k)}\mathrm{e}^{nxt}}{\prod_{i=0}^{r-1}\alpha_i^n\mathrm{e}^{nt}-1}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}\left(\alpha_{i-1}\mathrm{e}^t\right)^{s_i}=\frac{n^{rk}t^r2^{-r}(t)^{r(k-1)}2^{r(2-k)}\mathrm{e}^{nxt}}{\prod_{i=0}^{r-1}\alpha_i\mathrm{e}^t-1},$$

then, we have

$$\sum_{\ell=0}^{\infty}\frac{n^{\ell+r}\ell!}{(\ell+r)!}\frac{t^{\ell}}{\ell!}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}(\alpha_{i-1})^{s_i}M_{r+\ell}^{[0,r]}\left(x+\frac{\sum_{i=1}^{r}s_i}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)$$
$$=n^{rk}2^{-r}\sum_{\ell=0}^{\infty}M_{\ell}^{[0,r]}(nx;k-1;1,e,e;\bar{\alpha}_r)\frac{t^{\ell}}{\ell!}.$$

Equating coefficients of t^{ℓ} on both sides, yields (3.9). □

Theorem 3.6. *For* $(\bar{\boldsymbol{\alpha}}_r)^n=(\alpha_0^n,\alpha_1^n,\cdots,\alpha_{r-1}^n)$ *and* $(\bar{\boldsymbol{\alpha}}_r)^m=(\alpha_0^m,\alpha_1^m,\cdots,\alpha_{r-1}^m)$ *we have*

$$\begin{aligned}&n^{\ell}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}(\alpha_{i-1})^{ms_i}M_{\ell}^{[0,r]}\left(\frac{x}{n}+\frac{\left(\sum_{i=1}^{r}s_i\right)m}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)\\&=m^{-rk+\ell}n^{rk}\prod_{i=1}^{r}\sum_{p_i=0}^{m-1}(\alpha_{i-1})^{np_i}M_{\ell}^{[0,r]}\left(\frac{x}{m}+\frac{\left(\sum_{i=1}^{r}p_i\right)n}{m};k;1,e,e;(\bar{\alpha}_r)^m\right).\end{aligned}\tag{3.10}$$

$$\begin{aligned}&n^{\ell+r}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}(\alpha_{i-1})^{ms_i}M_{\ell+r}^{[0,r]}\left(\frac{x}{n}+\frac{\left(\sum_{i=1}^{r}s_i\right)m}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)\\&=\frac{m^{-r(k-1)+\ell}n^{rk}}{2^r}\prod_{i=1}^{r}\sum_{p_i=0}^{m-1}(\alpha_{i-1})^{np_i}M_{\ell}^{[0,r]}\left(\frac{x}{m}+\frac{\left(\sum_{i=1}^{r}p_i\right)n}{m};k-1;1,e,e;(\bar{\alpha}_r)^m\right).\end{aligned}\tag{3.11}$$

Proof. For the first equation and starting with (2.1), we get

$$\sum_{\ell=0}^{\infty}\frac{(nt)^{\ell}}{\ell!}\prod_{i=1}^{r}\sum_{s_i=0}^{n-1}(\alpha_{i-1})^{ms_i}M_{\ell}^{[0,r]}\left(\frac{x}{n}+\frac{\left(\sum_{i=1}^{r}s_i\right)m}{n};k;1,e,e;(\bar{\alpha}_r)^n\right)$$
$$=\frac{n^{rk}2^{r(1-k)}t^{rk}e^{xt}}{\prod_{i=0}^{r-1}(\alpha_i^n e^{nt}-1)}\frac{\prod_{i=0}^{r-1}(\alpha_i e^{nmt}-1)}{\prod_{i=0}^{r-1}(\alpha_i^m e^{mt}-1)}$$
$$=\frac{m^{-rk}n^{rk}2^{r(1-k)}t^{rk}m^{rk}e^{\left(\frac{x}{m}+\frac{\left(\sum_{i=1}^{r}p_i\right)n}{m}\right)mt}}{\prod_{i=0}^{r-1}(\alpha_i^m e^{mt}-1)}\prod_{i=1}^{r}\sum_{p_i=0}^{m-1}(\alpha_{i-1})^{np_i}$$
$$=m^{-rk}n^{rk}\sum_{\ell=0}^{\infty}\left(\prod_{i=1}^{r}\sum_{p_i=0}^{m-1}(\alpha_{i-1})^{np_i}m^{\ell}M_{\ell}^{[0,r]}\left(\frac{x}{m}+\frac{\left(\sum_{i=1}^{r}p_i\right)n}{m};k;1,e,e;(\bar{\alpha}_r)^m\right)\right)\frac{t^{\ell}}{\ell!}.$$

Equating the coefficients of t^{ℓ} on both sides, yields (3.10).

Also, It is not difficult to prove (3.11). □

4. Some Relations between $M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)$ and Other Polynomials and Numbers

In this section, we give some relationships between the polynomials $M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)$ and Laguerre polynomials, Jacobi polynomials, Hermite polynomials, generalized Stirling numbers of second kind, Stirling numbers and Bleimann-Butzer-hahn basic.

Theorem 4.1. *For* $\bar{\alpha}_r=(\alpha_0,\alpha_1,\cdots,\alpha_r)\in\mathbb{C}$, $(x;\bar{\alpha})_{\underline{\ell}}=(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{\ell-1})$ *and* $n,j\in\mathbb{N}_0$, *we have relationship*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{j=0}^{n}(x;\alpha)_{\underline{j}}\sum_{\ell=j}^{n}\binom{n}{n-\ell}(\ln c)^{\ell}S(\ell,j;\bar{\alpha})M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r) \tag{4.1}$$

between the new unification of Apostol-type polynomials and generalized Stirling numbers of second kind, see [19].

Proof. Using (3.4) and from definition of generalized Stirling numbers of second kind, we easily obtain (4.1). □

Theorem 4.2. *For* $\bar{\alpha}_r=(\alpha_0,\alpha_1,\cdots,\alpha_r)\in\mathbb{C}$, $(x)_{\underline{\ell}}=(x)(x-1)\cdots(x-\ell+1)$ *and* $n,j\in\mathbb{N}_0$, *we have the relationship*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{j=0}^{n}(x)_{\underline{j}}\sum_{\ell=j}^{n}\binom{n}{n-\ell}(\ln c)^{\ell}S(\ell,j)M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r) \tag{4.2}$$

between the new unification of Apostol-type polynomials and Stirling numbers of second kind.

Proof. Using (3.4) and from definition of Stirling numbers of second kind (see [20]), we easily obtain (4.2). □

Theorem 4.3. *The relationship*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{j=0}^{n}\sum_{\ell=j}^{n}(-1)^j\ell!\binom{n}{n-\ell}(\ln c)^{\ell}\binom{\ell+\alpha}{\ell-j}L_j^{(\alpha)}(x)M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r) \tag{4.3}$$

holds between the new unification of multiparameter Apostol-type polynomials and generalized Laguerre polynomials (see [7], *No.* (3), **Table 1**).

Proof. From (3.4) and substitute

$$x^{\ell}=\ell!\sum_{j=0}^{\ell}(-1)^j\binom{\ell+\alpha}{\ell-j}L_j^{\alpha}(x),$$

then we get (4.3). □

Theorem 4.4. *For* $(\alpha+\beta+j+1)_{\ell+1}=(\alpha+\beta+j+1)(\alpha+\beta+j+2)\cdots(\alpha+\beta+j+\ell+1)$. *The relationship*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{j=0}^{n}\sum_{\ell=j}^{n}(-1)^j\ell!\binom{n}{n-\ell}(\ln c)^{\ell}\binom{\ell+\alpha}{\ell-j}\frac{\alpha+\beta+2j+1}{(\alpha+\beta+j+1)_{\ell+1}}$$
$$P_j^{(\alpha,\beta)}(1-2x)M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r) \tag{4.4}$$

holds between the new unification of Apostol-type polynomials and Jacobi polynomials (see [21], p. 49, *Equation* (35)).

Proof. From (3.4) and substitute

$$x^{\ell}=\ell!\sum_{j=0}^{\ell}(-1)^j\binom{\ell+\alpha}{\ell-j}\frac{\alpha+\beta+2j+1}{(\alpha+\beta+j+1)_{\ell+1}}P_j^{(\alpha,\beta)}(1-2x),$$

then we get (4.4). □

Theorem 4.5. *The relationship*

$$M_n^{[m-1,r]}(x;k;a,b,c;\bar{\alpha}_r)=\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_{\ell=2j}^{n}2^{-\ell}\binom{n}{n-\ell}\binom{\ell}{2j}\frac{2j!}{j!}(\ln c)^{\ell}H_{\ell-2j}(x)M_{n-\ell}^{[m-1,r]}(k;a,b,c;\bar{\alpha}_r)\tag{4.5}$$

holds between the new unification of Apostol-type polynomials and Hermite polynomials (*see* [7], No. (1) **Table 1**).

Proof. From (3.4) and substitute

$$x^{\ell}=2^{-\ell}\sum_{j=0}^{\left[\frac{\ell}{2}\right]}\binom{\ell}{2j}\frac{2j!}{j!}H_{\ell-2j}(x),$$

then we get (4.5). □

Theorem 4.6. *When* $m=1$, $a=1$, $b=e$ *and* $c=e$ *in* (9) *and for* $\bar{\alpha}_r=(\alpha_0,\alpha_1,\cdots,\alpha_{r-1})$, $\overline{\boldsymbol{\alpha}^*}_r=\left(\frac{1}{\alpha_0},\frac{1}{\alpha_1},\cdots,\frac{1}{\alpha_{r-1}}\right)$, $\alpha_i\neq 0$, $i=0,1,\cdots,r-1$ *and* $\bar{\boldsymbol{\beta}}_m=(\beta_0,\beta_1,\cdots,\beta_{m-1})$, $\overline{\boldsymbol{\beta}^*}_m=\left(\frac{1}{\beta_0},\frac{1}{\beta_1},\cdots,\frac{1}{\beta_{m-1}}\right)$, $\beta_i\neq 0$, $i=0,1,\cdots,m-1$, *we have the following relationship*

$$M_n^{(r)}(x;k;\bar{\alpha}_r)=\frac{n!}{\prod_{i=0}^{r-1}\alpha_i}\sum_{m=r}^{\infty}\frac{2^{(1-k)(r-m)}\prod_{j=0}^{m-1}\beta_j}{(n+k(m-1))!}C\left(m,r;\overline{\alpha^*}_r;\overline{\beta^*}_m\right)M_{n+k(m-r)}^{(m)}\left(x;k;\bar{\beta}_m\right),\tag{4.6}$$

between the new unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, and $C\left(m,r;\overline{\alpha^*}_r;\overline{\beta^*}_m\right)$ (*the generalized Lah numbers*) (*see* [22]).

Proof. From [9], Equation (2.1),

$$\begin{aligned}
&\sum_{n=0}^{\infty}M_n^{(r)}(x;k;\bar{\alpha}_r)\frac{t^n}{n!}\\
&=\frac{t^{rk}2^{r(1-k)}\mathrm{e}^{xt}}{\prod_{i=1}^{r-1}(\alpha_i\mathrm{e}^t-1)}=\frac{t^{rk}2^{r(1-k)}\mathrm{e}^{xt}}{\prod_{i=1}^{r-1}\alpha_i}\frac{1}{\left(e^t;\overline{\alpha^*}\right)_{\underline{r}}}\\
&=\frac{t^{rk}2^{r(1-k)}\mathrm{e}^{xt}}{\prod_{i=1}^{r-1}\alpha_i}\sum_{m=r}^{\infty}C\left(m,r;\overline{\alpha^*}_r;\overline{\beta^*}_m\right)\frac{1}{\left(\mathrm{e}^t;\overline{\beta^*}\right)_{\underline{m}}}\\
&=\sum_{m=r}^{\infty}\frac{t^{k(r-m)}2^{(r-m)(1-k)}\prod_{j=1}^{m-1}\beta_j}{\prod_{i=1}^{r-1}\alpha_i}C\left(m,r;\overline{\alpha^*}_r;\overline{\beta^*}_m\right)\frac{t^{mk}2^{m(1-k)}\mathrm{e}^{xt}}{\left(\mathrm{e}^t;\overline{\beta^*}\right)_{\underline{m}}}\\
&=\sum_{n=0}^{\infty}\left(\sum_{m=r}^{\infty}\frac{n!2^{(1-k)(r-m)}\prod_{j=0}^{m-1}\beta_j}{(n+k(m-1))!\prod_{i=0}^{r-1}\alpha_i}C\left(m,r;\overline{\alpha^*}_r;\overline{\beta^*}_m\right)M_{n+k(m-r)}^{(m)}\left(x;k;\bar{\beta}_m\right)\right)\frac{t^n}{n!}.
\end{aligned}$$

Equating the coefficients of t^n on both sides, yields (4.6). □

Using No. 13 in **Table 1** (see [9]) and the definition of the unified Bernstein and Bleimann-Butzer-Hahn basis (see [23]),

$$\left(\frac{2^{1-k}x^kt^k}{(1+ax)^k}\right)^m\frac{1}{mk!}\mathrm{e}^{t\left(\frac{1+bx}{1+ax}\right)}=\sum_{n=0}^{\infty}p_n^{(a,b)}(x;k,m)\frac{t^n}{n!},\tag{4.7}$$

where $k,m\in\mathbb{Z}^+$, $a,b\in\mathbb{R}$, $t\in\mathbb{C}$, we obtain the following theorem.

Theorem 4.7. *For* $\alpha_i\neq 0, i=0,1,\cdots,r-1,$ *we have relationship*

$$P_n^{(a,b)}(x;k,r)=\frac{\prod_{i=0}^{r-1}\alpha_i}{rk!}\left(\frac{x}{1+ax}\right)^{rk}\sum_{j=0}^{r}s\left(r,j;\frac{1}{\bar{\alpha}_r}\right)\sum_{\ell=0}^{n}j^{n-\ell}\binom{n}{\ell}M_\ell^{(r)}\left(\frac{1+bx}{1+ax};k;\bar{\alpha}_r\right) \tag{4.8}$$

between the unified Bernstein and Bleimann-Butzer-Hahn basis, the new unified family of generalized Apostol-Bernoulli, Euler and Genocchi polynomials (*see* [9]) *and generalized Stirling numbers of first kind* (*see* [19]).

Proof. From (2.1) and (4.7) and with some elementary calculation, we easily obtain (4.8). □

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