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On the Derivation and Implementation of a Four Stage Harmonic Explicit Runge-Kutta Method*

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Abstract

In recent times, the derivation of Runge-Kutta methods based on averages other than the arithmetic mean is on the rise. In this paper, the authors propose a new version of explicit Runge-Kutta method, by introducing the harmonic mean as against the usual arithmetic averages in standard Runge-Kutta schemes.

Keywords

Explicit, Harmonic, Runge-Kutta, Autonomous

1. Introduction

During the last few decades, there has been a growing interest in problem solving systems based on the Runge-Kutta methods. Several methods have been developed using the idea different means such as the geometric mean, centroidal mean, harmonic mean, contra-harmonic mean and the heronian mean.

In previous papers [1] and [2], the authors presented a three stage method based on the harmonic mean and a multi-derivative method using the usual arithmetic mean respectively. Akanbi [3] developed a third-order method based on the geometric mean. In [4] and [5], the concept of the heronian mean was introduced. Evans and Yaacob [6] introduced a fourth-order method based on the harmonic mean while Yaacob and Sanugi [7] also developed a fourth-order method which is an embedded method based on the arithmetic and harmonic mean. Wazwaz [8] presented a comparison of modified Runge-Kutta methods based on varieties of means. Using the

*Four Stage Harmonic Runge-Kutta Method.

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definition of the harmonic mean, a fourth-order Runge-Kutta method is developed and implemented.

2. Derivation of the 4sHERK Method

The schemes introduced by [7] and [9] respectively are

$$y_{n+1}^{\text{HM}} = y_n + \frac{2}{3}h\left(\frac{k_1k_2}{k_1+k_2} + \frac{k_2k_3}{k_2+k_3} + \frac{k_3k_4}{k_3+k_4}\right) \tag{1}$$

where

$$k_1 = f\left(x_n, y_n\right)$$

$$k_2 = f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_n + \frac{1}{2}h, y_n - \frac{1}{2}hk_1 + \frac{5}{8}hk_2\right)$$

$$k_4 = f\left(x_n + h, y_n - \frac{1}{4}hk_1 + \frac{7}{20}hk_2 + \frac{9}{10}hk_3\right)$$

and

$$y_{n+1}^{\text{AHM}} = y_n + h\left[\frac{1}{6}k_2 + \frac{1}{6}k_3 + \frac{2}{3}\left(\frac{k_1k_2}{k_1+k_2}\right) + \frac{2}{3}\left(\frac{k_3k_4}{k_3+k_4}\right)\right] \tag{2}$$

where

$$k_1 = f\left(y_n\right)$$

$$k_2 = f\left(y_n + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(y_n - \frac{1}{2}hk_1 + \frac{5}{8}hk_2\right)$$

$$k_4 = f\left(y_n - \frac{1}{4}hk_1 + \frac{7}{20}hk_2 + \frac{9}{10}hk_3\right)$$

Scheme (2) was referred to as RK-HM-AM. Using the definition of harmonic mean, the following scheme is proposed in this paper:

$$y_{n+1} = y_n + h\Phi_H\left(y_n; h\right) \tag{3}$$

where,

$$\Phi_H\left(y_n; h\right) = \frac{4k_1k_2k_3k_4}{k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4} \tag{4}$$

$$k_1 = f\left(y_n\right)$$

$$k_2 = f\left(y_n + b_{21}hk_1\right) \tag{5}$$

$$k_3 = f\left(y_n + b_{31}hk_1 + b_{32}hk_2\right) \tag{6}$$

$$k_4 = f\left(y_n + b_{41}hk_1 + b_{42}hk_2 + b_{43}hk_3\right) \tag{7}$$

where b_{21}, b_{31}, b_{32}, b_{41}, b_{42} and b_{43} are constants to be determined.

The expansion of k_2, k_3 and k_4 as defined above give

$$k_2 = f + fhb_{21} f_y + \frac{1}{2} f^2 h^2 b_{21}^2 f_{yy} + \frac{1}{6} f^3 h^3 b_{21}^3 f_{yyy} + O\left(h^4\right) \tag{8}$$

$$\begin{aligned} k_3 = {} & f + fh\left(b_{31} + b_{32}\right) f_y + h^2 \left(fb_{21}b_{32} f_y^2 + \frac{1}{2} f^2 \left(b_{31} + b_{32}\right)^2 f_{yy} \right) \\ & + h^3 \left(f^2 \left(\frac{1}{2} b_{21}^2 b_{32} + b_{21} b_{32} \left(b_{31} + b_{32}\right) \right) f_y f_{yy} + \frac{1}{6} f^3 \left(b_{31} + b_{32}\right)^3 f_{yyy} \right) + O\left(h^4\right), \end{aligned} \tag{9}$$

$$\begin{aligned} k_4 = {} & f + fh\left(b_{41} + b_{42} + b_{43}\right) f_y + h^2 \Big(f \left(b_{21}b_{42} + b_{31}b_{43} + b_{32}b_{43}\right) f_y^2 \\ & + f^2 \left(\frac{b_{41}^2}{2} + b_{41}b_{42} + \frac{b_{42}^2}{2} + b_{41}b_{43} + b_{42}b_{43} + \frac{b_{43}^2}{2} \right) f_{yy} \Big) \\ & + h^3 \Big(fb_{21}b_{32}b_{43} f_y^3 + f^2 \Big(\frac{1}{2} b_{21}^2 b_{42} + \frac{1}{2} b_{31}^2 b_{43} + \frac{1}{2} b_{32}^2 b_{43} \\ & + b_{21}b_{42}\left(b_{41} + b_{42} + b_{43}\right) + b_{32}b_{43}\left(b_{41} + b_{42} + b_{43}\right) \\ & + b_{31}b_{43}\left(b_{32} + b_{41} + b_{42} + b_{43}\right)\Big) f_y f_{yy} + \frac{1}{6} f^3 \left(b_{41} + b_{42} + b_{43}\right)^3 f_{yyy} \Big) + O\left(h^4\right), \end{aligned} \tag{10}$$

Substituting (8), (9) and (10) into (4) and simplifying the resulting expression using MATHEMATICA (version 8.0.1) package, the coefficients of the powers of h in (4) are compared with that of the Taylors' expansion of $\Phi_H\left(y_n;h\right)$ and upon solving the resulting system of non-linear equations we have

$$b_{21} = \frac{1}{2}; \quad b_{31} = 0; \quad b_{32} = 1; \quad b_{41} = 0; \quad b_{42} = 0; \quad b_{43} = \frac{1}{2}; \tag{11}$$

Thus, the incremental function (4) of the proposed scheme is

$$\Phi_H\left(y_n;h\right) = f + \frac{1}{2} fhf_y + h^2 \left(\frac{ff_y^2}{8} + \frac{3f^2 f_{yy}}{16} \right) + h^3 \left(\frac{5}{32} f^2 f_y f_y + \frac{5f^3 f_{yyy}}{96} \right) \tag{12}$$

and the proposed scheme (3) is

$$y_{n+1} = y_n + 4h \frac{k_1k_2k_3k_4}{k_1k_2k_3 + k_1k_2k_4 + k_1k_3k_4 + k_2k_3k_4} \tag{13}$$

where

$$k_1 = f\left(y_n\right)$$

$$k_2 = f\left(y_n + \frac{1}{2}hk_1\right) \tag{14}$$

$$k_3 = f\left(y_n + hk_2\right) \tag{15}$$

$$k_4 = f\left(y_n + \frac{1}{2}hk_3\right) \tag{16}$$

3. Stability of the 4sHERK Method

For the analysis of the absolute stability of the proposed 4sHERK scheme, the scalar test problem $y' = \lambda y$ with solution $y = e^{\lambda y}$ is used, where λ is a complex variable (see [10]). With the above test problem, we have

$$k_1 = f\left(y_n\right) = \lambda y_n \tag{17}$$

$$k_2 = f\left(y_n + \frac{1}{2}hk_1\right) = \lambda y_n \left(1 + \frac{h\lambda}{2}\right) \tag{18}$$

$$k_3 = f\left(y_n + hk_2\right) = \lambda y_n\left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) \tag{19}$$

$$k_4 = f\left(y_n + \frac{1}{2}hk_3\right) = \lambda y_n\left(1 + \frac{h\lambda}{2} + \frac{h^2\lambda^2}{2} + \frac{h^3\lambda^3}{4}\right) \tag{20}$$

Substituting (17)-(20) in (3) and simplifying the resulting expression results in,

$$y_{n+1} = y_n + h\lambda y_n + \frac{1}{2}h^2\lambda^2 y_n + \frac{1}{8}h^3\lambda^3 y_n - \frac{1}{64}h^5\lambda^5 y_n \tag{21}$$

Letting $z = \lambda h$ and evaluating $\frac{y_{n+1}}{y_n}$ from (21), the stability polynomial of the proposed scheme is obtained as

$$R(z) = \frac{y_{n+1}}{y_n} = 1 + z + \frac{1}{2}z^2 + \frac{1}{8}z^3 - \frac{1}{64}z^5 + O\left(z^6\right) \tag{22}$$

The absolute stability region of the 4sHERK scheme is given in **Figure 1**.

4. Error Estimation

Definition: The local truncation error at x_{n+1} of the explicit one step method (3) is defined to be T_{n+1} where

$$T_{n+1} = y\left(x_{n+1}\right) - y\left(x_n\right) - h\Phi_H\left(y\left(x_n\right);h\right)$$

And $y\left(x_n\right)$ is the theoretical solution (See [10]).

Using the above definition together with (12), the local truncation error (LTE) of the proposed scheme is given as

$$\begin{aligned}\mathrm{LTE}\left(4\mathrm{sHERK}\right) &= y\left(x_n + h\right) - y_{n+1}\\ &= h^4\left(-\frac{1}{64}ff_y^4 + \frac{1}{16}f^2 f_y^2 f_{yy} + f^3\left(\frac{7f_{yy}^2}{256} + \frac{f_y f_{yyy}}{16}\right) + \frac{17f^4 f_{yyyy}}{1536}\right) + O\left(h^5\right),\end{aligned} \tag{23}$$

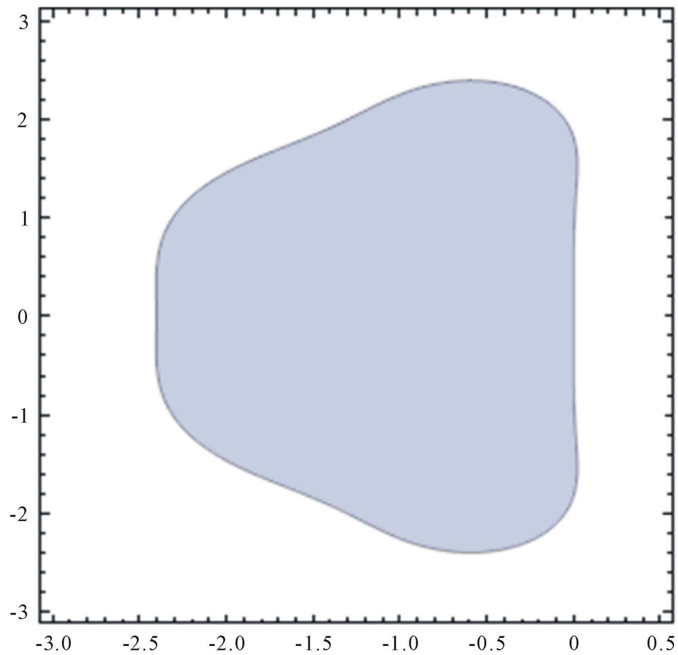

Figure 1. Absolute stability region of the 4sHERK method.

where $y(x_n + h)$ is obtained by Taylor series expansion.

5. Numerical Experiments

Consider the IVP

$$y' = \frac{1}{y}, \quad y(0) = 1 \tag{24}$$

with the theoretical solution

$$y = \sqrt{2x+1} \tag{25}$$

Table 1. h =0.125, $y'(x) = \dfrac{1}{y(x)}$, $y(0) = 1$, exact solution: $y(x) = \sqrt{2x+1}$.

x	Exact Sol.	RK-4	RK-HM-AM [4]	RK-HM [6]	4sHERK	RK-4 Error	RK-HM-AM [4] Error	RK-HM [6] Error	4sHERK Error
0.125	1.11803399	1.11803441	1.11803365	1.11803347	1.11803399	0.42308247e−6	0.3380746e−6	0.52268107e−6	0.37325369e−8
0.25	1.22474487	1.22474543	1.22474443	1.22474419	1.22474487	0.55362188e−6	0.44595043e−6	0.68606666e−6	0.44036872e−8
0.375	1.32287566	1.32287625	1.32287518	1.32287492	1.32287565	0.59022189e−6	0.47774995e−6	0.7328127e−6	0.44098678e−8
0.5	1.41421356	1.41421415	1.41421308	1.41421283	1.41421356	0.59242193e−6	0.48102403e−6	0.73644671e−6	0.42553943e−8
0.625	1.5	1.50000058	1.49999953	1.49999928	1.5	0.58129728e−6	0.47297202e−6	0.72321365e−6	0.4069489e−8
0.75	1.58113883	1.5811394	1.58113837	1.58113813	1.58113883	0.56516855e−6	0.46050983e−6	0.70355079e−6	0.38884294e−8
0.875	1.6583124	1.65831294	1.65831195	1.65831171	1.65831239	0.54755259e−6	0.44661313e−6	0.68190159e−6	0.37219154e−8
	1.73205081	1.73205134	1.73205037	1.73205015	1.7320508	0.52998364e−6	0.43260686e−6	0.66022089e−6	0.35714376e−8
0.125	1.80277564	1.80277615	1.80277522	1.802775	1.80277563	0.5131226e−6	0.41907842e−6	0.63936096e−6	0.34359544e−8
0.25	1.87082869	1.87082919	1.87082829	1.87082807	1.87082869	0.49722869e−6	0.4062709e−6	0.61966383e−6	0.33137697e−8
0.375	1.93649167	1.93649216	1.93649128	1.93649107	1.93649167	0.48237238e−6	0.39426268e−6	0.60123001e−6	0.32031635e−8
0.5	2.0	2.00000047	1.99999962	1.99999942	2.0	0.46853586e−6	0.38305324e−6	0.58404588e−6	0.31025875e−8

Table 2. h = 0.1, $y'(x) = \dfrac{1}{y(x)}$, $y(0) = 1$, exact solution: $y(x) = \sqrt{2x+1}$.

x	Exact Sol.	RK-4	RK-HM-AM [4]	RK-HM [6]	4sHERK	RK-4 Error	RK-HM-AM [4] Error	RK-HM [6] Error	4sHERK Error
0.1	1.09544512	1.09544526	1.09544499	1.09544493	1.09544511	0.14972954e−6	0.12283314e−6	0.18686867e−6	0.89117402e−9
0.2	1.18321596	1.18321616	1.18321578	1.1832157	1.18321596	0.20809082e−6	0.17175178e−6	0.26033694e−6	0.1122773e−8
0.3	1.26491106	1.26491129	1.26491087	1.26491077	1.26491106	0.23071443e−6	0.19118572e−6	0.28910566e−6	0.11668382e−8
0.4	1.34164079	1.34164102	1.34164059	1.34164049	1.34164079	0.23787374e−6	0.19765556e−6	0.29840694e−6	0.11515036e−8
0.5	1.41421356	1.4142138	1.41421336	1.41421326	1.41421356	0.23792188e−6	0.19807497e−6	0.29870149e−6	0.11172532e−8
0.6	1.4832397	1.48323993	1.4832395	1.4832394	1.4832397	0.23461819e−6	0.19559589e−6	0.29472188e−6	0.10781778e−8
0.7	1.54919334	1.54919357	1.54919315	1.54919305	1.54919334	0.22976557e−6	0.19174751e−6	0.28874859e−6	0.10394097e−8
0.8	1.61245155	1.61245177	1.61245136	1.61245127	1.61245155	0.22426742e−6	0.18730473e−6	0.2819297e−6	0.10027719e−8
0.9	1.67332005	1.67332027	1.67331987	1.67331978	1.67332005	0.21858829e−6	0.18267092e−6	0.27485859e−6	0.96880082e−9
0.0	1.73205081	1.73205102	1.73205063	1.73205054	1.73205081	0.21296863e−6	0.17805796e−6	0.26784435e−6	0.9375225e−9

Table 3. $h = 0.01$, $y'(x) = \frac{1}{y(x)}$, $y(0) = 1$, exact solution: $y(x) = \sqrt{2x+1}$.

x	Exact Sol.	RK-4	RK-HM-AM [4]	RK-HM [6]	4sHERK	RK-4 Error	RK-HM-AM [4] Error	RK-HM [6] Error	4sHERK Error
0.01	1.00995049	1.00995049	1.00995049	1.00995049	1.00995049	0.20121682e−11	0.18316459e−11	0.26254554e−11	0.2220446e−15
0.02	1.0198039	1.0198039	1.0198039	1.0198039	1.0198039	0.38344883e−11	0.34909853e−11	0.50035531e−11	0.44408921e−15
0.03	1.02956301	1.02956301	1.02956301	1.02956301	1.02956301	0.54869442e−11	0.49957816e−11	0.71600503e−11	0.44408921e−15
0.04	1.03923048	1.03923048	1.03923048	1.03923048	1.03923048	0.69868555e−11	0.63624661e−11	0.91180397e−11	0.66613381e−15
0.05	1.04880885	1.04880885	1.04880885	1.04880885	1.04880885	0.83497653e−11	0.76043616e−11	0.10897283e−10	0.66613381e−15
0.06	1.05830052	1.05830052	1.05830052	1.05830052	1.05830052	0.95894404e−11	0.87341245e−11	0.12515544e−10	0.66613381e−15
0.07	1.06770783	1.06770783	1.06770783	1.06770783	1.06770783	0.10717871e−10	0.97626351e−11	0.13988588e−10	0.66613381e−15
0.08	1.07703296	1.07703296	1.07703296	1.07703296	1.07703296	0.11745716e−10	0.10699663e−10	0.15330626e−10	0.66613381e−15
0.09	1.08627805	1.08627805	1.08627805	1.08627805	1.08627805	0.12682522e−10	0.11553869e−10	0.16553869e−10	0.66613381e−15
0.1	1.09544512	1.09544512	1.09544511	1.09544511	1.09544512	0.13536727e−10	0.12333246e−10	0.17669644e−10	0.66613381e−15

We apply the new 4sHERK method (13) to the above IVP and the results obtained are compared with the classical 4-stage fourth-order Runge-Kutta method and the methods of [6] and [4].

The results generated by the newly derived scheme in this paper evidently proved the extent of accuracy of the scheme in comparison with the other methods.

6. Conclusion

Evidently, the newly derived scheme is more accurate as seen from the computational results presented in **Table 1**, **Table 2** and **Table 3**, since its absolute error is the least of all the methods presented in this paper. It therefore follows that the scheme is quite efficient. We therefore conclude that the 4sHERK method proposed is reliable, stable and with high accuracy in computation.

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