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Necessary Conditions for the Application of Moving Average Process of Order Three

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Abstract

Invertibility is one of the desirable properties of moving average processes. This study derives consequences of the invertibility condition on the parameters of a moving average process of order three. The study also establishes the intervals for the first three autocorrelation coefficients of the moving average process of order three for the purpose of distinguishing between the process and any other process (linear or nonlinear) with similar autocorrelation structure. For an invertible moving average process of order three, the intervals obtained are $\frac{-1-\sqrt{5}}{4} < \rho_1 < \frac{1-\sqrt{5}}{4}$, $-0.5 < \rho_2 < 0.5$ and $-0.5 < \rho_1 < 0.5$.

Keywords

Moving Average Process of Order Three, Characteristic Equation, Invertibility Condition, Autocorrelation Coefficient, Second Derivative Test

1. Introduction

Moving average processes (models) constitute a special class of linear time series models. A moving average process of order q ($\mathrm{MA}(q)$ process) is of the form:

$$X_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q} + e_q \tag{1.1}$$

where $\theta_1, \theta_2, \cdots, \theta_q$ are real constants and e_t, $t \in Z$ is a sequence of independent and identically distributed random variables with zero mean and constant variance. These processes have been widely used to model

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time series data from many fields [1]-[3]. The model in (1.1) is always stationary. Hence, a required condition for the use of the moving average process is that it is invertible. Let $B^m e_t = e_{t-m}$, then the model in (1.1) is invertible if the roots of the characteristic equation

$$1+\theta_1 B+\theta_2 B^2+\cdots+\theta_q B^q=0 \tag{1.2}$$

lie outside the unit circle. The invertibility conditions of the first order and second order moving average models have been derived [4] [5].

Ref. [6] used a moving average process of order three (MA (3) process) in his simulation study. Though, higher order moving average processes have been used to model time series data, not much has been said about the properties of their autocorrelation functions. This study focuses on the invertibility condition of an MA (3) process. Consideration is also given to the properties of its autocorrelation coefficients of an invertible moving average process of order three.

2. Consequence of Invertibility Condition on the Parameters of an MA (3) Process

For $q=3$, the following moving average process of order 3 is obtained from (1.1):

$$X_t=\theta_1 e_{t-1}+\theta_2 e_{t-2}+\theta_3 e_{t-3}+e_t \tag{2.1}$$

The characteristic equation corresponding to (2.1) is given by

$$1+\theta_1 B+\theta_2 B^2+\theta_3 B^3=0 \tag{2.2}$$

Dividing (2.2) by θ_3 yields

$$B^3+\frac{\theta_2}{\theta_3}B^2+\frac{\theta_1}{\theta_3}B+\frac{1}{\theta_3}=0 \tag{2.3}$$

It is important to know that (2.2) is a cubic equation. Detailed information on how to solve cubic equations can be found in [7] [8] among others. It has been a common tradition to consider the nature of the roots of a characteristic equation while determining the invertibility condition of a time series model [9]. As a cubic equation, (2.2) may have three distinct real roots, one real root and two complex roots, two real equal roots or three real equal roots. The nature of the roots of (2.2) is determined with the help of the discriminant [8]

$$D=D_1^2-D_2^3 \tag{2.4}$$

where

$$D_1=\frac{2\left(\frac{\theta_2}{\theta_3}\right)^3-9\left(\frac{\theta_2}{\theta_3}\right)\left(\frac{\theta_1}{\theta_3}\right)+27\left(\frac{1}{\theta_3}\right)}{54} \tag{2.5}$$

and

$$D_2=\frac{\left(\frac{\theta_2}{\theta_3}\right)^2-3\left(\frac{\theta_1}{\theta_3}\right)}{9} \tag{2.6}$$

If $D<0$, (2.2) has the following distinct roots [7]

$$x_1=-2\sqrt{D_2}\cos\left(\frac{\theta}{3}\right)-\frac{\theta_2}{3}, \tag{2.7}$$

$$x_2=-2\sqrt{D_2}\cos\left(\frac{\theta+2\pi}{3}\right)-\frac{\theta_2}{3}, \tag{2.8}$$

and

$$x_3 = -2\sqrt{D_2}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{\theta_2}{3}. \tag{2.9}$$

where θ is measured in radians and $\theta = \cos^{-1}\left(\frac{D_1}{\sqrt{D_2^3}}\right)$.

When $D > 0$, (2.2) has only real root given by [1] as

$$x_1 = \sqrt[3]{-D_1 + \sqrt{D}} + \sqrt[3]{-D_1 - \sqrt{D}} - \frac{\theta_2}{3} \tag{2.10}$$

The other roots are [8]

$$x_2,\ x_3 = \frac{-(ax_1 + b) \pm \sqrt{(ax_1 + b)^2 - 4a(ax_1^2 + bx_1 + c)}}{2a} \tag{2.11}$$

If $D_1 \neq 0$, $D_2 \neq 0$ and $D_1^2 = D_2^3$, then $D = 0$ and (2.2) has two equal roots. The roots of (2.2) in this case, are the same as (2.7), (2.8) and (2.9). For $D = 0$ and $D_1 = D_2 = 0$, (2.2) has three real equal roots. Each of these roots is given by [8] as

$$x = \frac{-\theta_2}{3\theta_3} \tag{2.12}$$

For (2.1) to be invertible, the roots of (2.2) are all expected to lie outside the unit circle and $|\theta_3| < 1$. In the following theorem, the invertibility conditions of an MA (3) process are given subject to the condition that the corresponding characteristic equation has three real equal roots.

Theorem 1. If the characteristic equation $1 + \theta_1 B + \theta_2 B^2 + \theta_3 B^3 = 0$ has three real equal roots, then the moving average process of order three $X_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + \theta_3 e_{t-3} + e_t$ is invertible if

$$\theta_2 - 3\theta_3 > 0,\ \ \theta_2 + 3\theta_3 < 0 \ \text{ and } \ |\theta_3| < 1.$$

Proof

For invertibility, we expect each of the three real equal roots to lie outside the unit circle. Thus,

$$\left|\frac{-\theta_2}{3\theta_3}\right| > 1 \Rightarrow \frac{-\theta_2}{3\theta_3} < -1 \ \text{ or } \ \frac{-\theta_2}{3\theta_3} > 1$$

Solving the inequality $\frac{-\theta_2}{3\theta_3} < -1$, we obtain

$$\theta_2 - 3\theta_3 > 0$$

For $\frac{-\theta_2}{3\theta_3} > 1$, we have

$$\theta_2 + 3\theta_3 < 0$$

Since each of the roots lie outside the unit circle, the absolute value of their product must therefore be greater than one. Hence,

$$|\theta_3| < 1$$

This completes the proof.

The invertibility region of a moving average of order three with equal roots of the characteristic Equation (2.2) is enclosed by triangle OAB in **Figure 1**.

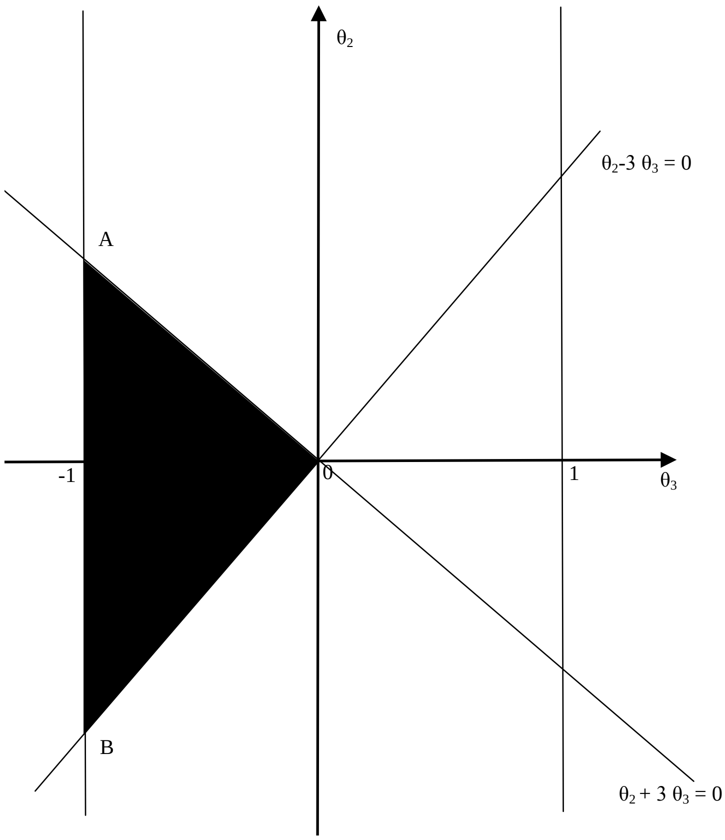

Figure 1. Invertibility region of an MA (3) process when the characteristic equation has three real equal roots.

3. Identification of Moving Average Process

Model identification is a crucial aspect of time series analysis. A common practice is to examine the structures of the autocorrelation function (ACF) and partial autocorrelation function (PACF) of a given time series. In this regard, a time series is said to follow a moving average process of order q if its associated autocorrelation function cut off after lag q and the corresponding partial autocorrelation function decays exponentially [10]. Authors using this method, believe that each process has unique ACF representation. However, the existence of similar autocorrelation structures between moving average process and pure diagonal bilinear time series process of the same order makes it difficult to identify a moving average process based on the pattern of its ACF. Furthermore, a careful look at the autocorrelation function of the square of a time series can help one determine if the series follows a moving average process. If the series can be generated by a moving average process, then its square follows a moving average process of the same order [11] [12]. The conditions under which we use the autocorrelation function to distinguish among processes behaving like moving average processes of order one and two have been determined by [13] [14] respectively. These conditions are all defined in terms of the extreme values of autocorrelation coefficients of the processes.

4. Intervals for Autocorrelation Coefficients of a Moving Average Process of Order Three

As stated in Section 3, knowledge of the extreme values of the autocorrelation coefficient of a moving average process of a particular order can enable us ensure proper identification of the process. It has been observed that for a moving average process of order one, $-0.5 \le \rho_1 \le 0.5$ [15] while for a moving average process of order two $-\frac{\sqrt{2}}{2} \le \rho_1 \le \frac{\sqrt{2}}{2}$ and $-0.5 \le \rho_2 \le 0.5$ [5]. In order to generalize about the range of values of ρ_q for a moving average process of order q, it is worthwhile to determine the range values of ρ_3 for a moving average process of order three. The model in (2.1) has the following autocorrelation function [10]:

$$\rho_k = \begin{cases} 1, & k = 0 \\ \dfrac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2}, & k = \pm 1 \\ \dfrac{\theta_2 + \theta_1\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2}, & k = \pm 2 \\ \dfrac{\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2}, & k = \pm 3 \\ 0, & k \neq \pm 1, \pm 2, \pm 3 \end{cases} \tag{4.1}$$

We can deduce from (4.1) that the autocorrelation function at lag one of the MA (3) process is

$$\rho_1 = \frac{\theta_1 + \theta_1\theta_2 + \theta_2\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} \tag{4.2}$$

Using the Scientific Note Book, the minimum and maximum values of ρ_1 are found to be $\frac{-1-\sqrt{5}}{4}$ and $\frac{1-\sqrt{5}}{4}$ respectively. For the autocorrelation function at lag two, we have

$$\rho_2 = \frac{\theta_2 + \theta_1\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} \tag{4.3}$$

The extreme values of ρ_2 are equally obtained with the help of the Scientific Note Book. To this effect, ρ_2 has a minimum value of −0.5 and a maximum value of 0.5.

From (4.1), we obtain

$$\rho_3 = \frac{\theta_3}{1+\theta_1^2+\theta_2^2+\theta_3^2} \tag{4.4}$$

Based on the result obtained from the Scientific Notebook, ρ_3 has a minimum value of −0.5 and a maximum value of 0.5. However, the intervals for ρ_3 can easily be obtained analytically and this result is generalized in Theorem 2 for ρ_q of the MA (q) process.

The partial derivatives of ρ_3 with respect to θ_1, θ_2 and θ_3 are

$$\frac{\partial\rho_3}{\partial\theta_1} = \frac{-2\theta_1\theta_3}{\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)^2} \tag{4.5}$$

$$\frac{\partial\rho_3}{\partial\theta_2} = \frac{-2\theta_2\theta_3}{\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)^2} \tag{4.6}$$

$$\frac{\partial\rho_3}{\partial\theta_3} = \frac{1+\theta_1^2+\theta_2^2-\theta_3^2}{\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)^2} \tag{4.7}$$

The critical points of ρ_3 occurs when $\frac{\partial\rho_3}{\partial\theta_i} =$, $i = 1,2,3$. Equating each of the partial derivatives in (4.5), (4.6) and (4.7) to zero, we obtain

$$\theta_1\theta_3 = 0 \tag{4.8}$$

$$\theta_2\theta_3 = 0 \tag{4.9}$$

$$1+\theta_1^2+\theta_2^2-\theta_3^2 = 0 \tag{4.10}$$

From (4.10), we have

$$\theta_3 = \pm\sqrt{1+\theta_1^2+\theta_2^2} \tag{4.11}$$

Using (4.8), we obtain

$$\theta_1 = 0 \tag{4.12}$$

or

$$\theta_3 = 0 \tag{4.13}$$

Substituting $\theta_1 = 0$ into (4.11) yields

$$\theta_3 = \pm\sqrt{1+\theta_2^2} \tag{4.14}$$

For $\theta_3 = -\sqrt{1+\theta_2^2}$, (4.9) becomes

$$\theta_2\left(\sqrt{1+\theta_2^2}\right) = 0$$

$$\theta_2^2\left(1+\theta_2^2\right) = 0$$

$$\theta_2 = 0 \quad \text{or} \quad \theta_2 = \pm\sqrt{-1} \tag{4.15}$$

If we also substitute $\theta_3 = \sqrt{1+\theta_2^2}$ into (4.9), we obtain

$$\theta_2 = 0 \quad \text{or} \quad \theta_2 = \pm\sqrt{-1} \tag{4.16}$$

When we substitute $\theta_1 = 0$ and $\theta_2 = 0$ into (4.11), we have $\theta_3 = \pm 1$. It is also clear that if $\theta_1 = 0$ and $\theta_2 = -\sqrt{-1}$, then $\theta_3 = 0$. Similar result is obtained when $\theta_1 = 0$ and $\theta_2 = \sqrt{-1}$.

Hence, the critical points of ρ_3 are $(0,0,-1)$, $(0,0,1)$, $\left(0,-\sqrt{-1},0\right)$ and $\left(0,\sqrt{-1},0\right)$.

The minimum and maximum values of a function occur at it critical points. To determine which of the critical points is a local minimum, local maximum or a saddle point, we shall apply the second derivative test. The second derivative test for critical points of a function of three variables $\rho_3 = f(x,y,z)$ focuses on the Hessian matrix:

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} \tag{4.17}$$

where

$$f_{xx} = \frac{\partial^2\rho_3}{\partial\theta_1^2} = \frac{-2\theta_3\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)+8\theta_1\theta_3^2}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.18}$$

$$f_{xy} = \frac{\partial^2\rho_3}{\partial\theta_1\partial\theta_2} = \frac{-8\theta_1\theta_2\theta_3}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.19}$$

$$f_{xz} = \frac{\partial^2\rho_3}{\partial\theta_1\partial\theta_3} = \frac{-2\theta_1\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)+8\theta_1\theta_3^2}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.20}$$

$$f_{yy} = \frac{-2\theta_3\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)+8\theta_2^2\theta_3}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.21}$$

$$f_{yz} = \frac{-2\theta_2\left(1+\theta_1^2+\theta_2^2+\theta_3^2\right)+8\theta_2\theta_3^2}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.22}$$

$$f_{zz} = \frac{-2\theta_3\left(1+\theta_1^2\theta_2^2\theta_3^2\right)-4\theta_3\left(1+\theta_1^2+\theta_2^2-\theta_3^2\right)}{\left(1+\theta_1^2\theta_2^2\theta_3^2\right)^3} \tag{4.23}$$

Let (a,b,c) be a critical point of $\rho_3 = f(x,y,z)$. Then (a,b,c) is called a local minimum point if at (a,b,c), $\Delta_1 = f_{xx} > 0$, $\Delta_2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0$ and $\Delta_3 = |H| > 0$ [16]. If $f_{xx} < 0$, $\Delta_2 > 0$ and $\Delta_3 < 0$ at (a,b,c), then (a,b,c) represents a local maximum.

A critical point that is neither a local minimum nor a local maximum is called a saddle point.

Though ρ_3 has four critical points, it is not defined at $\left(0,-\sqrt{-1},0\right)$ and $\left(0,\sqrt{-1},0\right)$. We then focus on the classification of the two remaining critical points.

At $(0,0,-1)$

$$H = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Hence, $\Delta_1 = \frac{1}{2} > 0$, $\Delta_2 = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{4} > 0$ and $\Delta_3 = \begin{vmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{8} > 0$.

Therefore, $(0,0,-1)$ is a local minimum. The value of ρ_3 at this point is -0.5.

For the critical points $(0,0,1)$, we have

$$H = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

Consequently,

$$\Delta_1 = -\frac{1}{2} < 0,$$

$$\Delta_2 = \begin{vmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{vmatrix} = \frac{1}{4} > 0$$

and

$$\Delta_3 = \begin{vmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{vmatrix} = -\frac{1}{8} < 0.$$

We therefore conclude that $(0,0,1)$ is a local maximum. The maximum value of ρ_3 obtained at $(0,0,1)$ is 0.5.

We can deduce from the result in this section and other previous works that for MA (1) process $|\rho_1| \le 0.5$, while for MA (2) process and MA (3) process $|\rho_2| \le 0.5$ and $|\rho_3| \le 0.5$ respectively.

In what follows, we establish the bounds for ρ_q, where q is order of the moving average process.

Theorem 2.

Let $X_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q} + e_q$ be an MA (q) process. Then, $|\rho_q| \le 0.5$.

Proof

It is easily seen that for the MA (q) process,

$$\rho_q = \frac{\theta_q}{1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2}$$

Partial derivatives of ρ_q with respect to $\rho_q = \theta_1, \theta_2, \cdots, \theta_q$ are as follows

$$\frac{\partial \rho_q}{\partial \theta_1} = \frac{-2\theta_1\theta_q}{\left(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2\right)^2},$$
$$\frac{\partial \rho_q}{\partial \theta_2} = \frac{-2\theta_2\theta_q}{\left(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2\right)^2},$$
$$\vdots$$
$$\frac{\partial \rho_q}{\partial \theta_{q-1}} = \frac{-2\theta_{q-1}\theta_q}{\left(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2\right)^2},$$
$$\frac{\partial \rho_q}{\partial \theta_q} = \frac{1+\theta_1^2+\theta_2^2+\cdots+\theta_{q-1}^2-\theta_q^2}{\left(1+\theta_1^2+\theta_2^2+\cdots+\theta_q^2\right)^2}.$$

Equating each of the partial derivatives to zero yields

$$\begin{aligned} -2\theta_1\theta_q &= 0, \\ -2\theta_2\theta_q &= 0, \\ &\vdots \\ -2\theta_{q-1}\theta_q &= 0, \\ 1+\theta_1^2+\theta_2^2+\cdots+\theta_{q-1}^2-\theta_q^2 &= 0. \end{aligned} \tag{4.24}$$

From (4.24), we obtain

$$\theta_q = \pm\sqrt{1+\theta_1^2+\theta_2^2+\cdots-\theta_{q-1}^2} \tag{4.25}$$

Since $\theta_q \ne 0$ for an MA (q) process, it is obvious that the $q-1$ equations preceding (4.24) are only satisfied if $\theta_1 = \theta_2 = \cdots = \theta_{q-1} = 0$. Substituting $\theta_1 = \theta_2 = \cdots = \theta_{q-1} = 0$ into (4.25) leads to $\theta_q = \pm 1$. The two critical points of ρ_q are then $(0,0,0,\cdots,-1)$ and $(0,0,0,\cdots,1)$.

At $(0,0,0,\cdots,-1)$, $\rho_q = -0.5$ while at $(0,0,0,\cdots,1)$, $\rho_q = 0.5$. It then follows that $|\rho_q| \le 0.5$.

Remark: For an invertible MA (3) process, $|\theta_3| < 1$. Hence, $\frac{-1-\sqrt{5}}{4} < \rho_1 < \frac{1-\sqrt{5}}{4}$, $-0.5 < \rho_2 < 0.5$ and $-0.5 < \rho_1 < 0.5$.

5. Conclusion

We have established necessary conditions for the parameters of an invertible MA (3) process. When the characteristic equation has three real equal roots, the conditions are $\theta_2 - 3\theta_3 > 0$, $\theta_2 + 3\theta_3 < 0$ and $|\theta_3| < 1$. Also the intervals for the autocorrelation coefficients of an invertible moving average process of order three are established. These are $\frac{-1-\sqrt{5}}{4} < \rho_1 < \frac{1-\sqrt{5}}{4}$, $-0.5 < \rho_2 < 0.5$ and $-0.5 < \rho_1 < 0.5$. It is also noteworthy that the condition on ρ_3 for an invertible MA (3) process is generalized for ρ_q of the invertible MA (q) process. That is for the invertible MA (q) process, $|\rho_q| < 0.5$. These results can now be used to compare other linear and nonlinear processes that have similar autocorrelation structures with the MA (3) process.

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