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Conventional and Added-Order Proportional Nonlinear Integral Observers

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Abstract

In this paper, a kind of fire new nonlinear integrator and integral action is proposed. Consequently, a conventional Proportional Nonlinear Integral (P_NI) observer and two kinds of added-order P_NI observers are developed to deal with the uncertain nonlinear system. The conditions on the observer gains to ensure the estimated error to be ultimate boundness, which shrinks to zero as the states and control inputs converge to the equilibrium point, are provided. This means that if the observed system is asymptotically stable, the estimated error dynamics is asymptotically stable, too. Moreover, the highlight point of this paper is that the design of nonlinear integral observer is achieved by linear system theory. Simulation results showed that under the normal and perturbed cases, the pure added-order P_NI observer can effectively deal with the uncertain nonlinearities on both the system dynamics and measured outputs.

Keywords

State Estimation, PI Observer, Nonlinear Integral Observer, Added-Order Observer, Nonlinear Integrator, Nonlinear Integral Action, Model-Based Observer

1. Introduction

State observer design plays an essential role in the design of control system. Compared with most type of observers, the Proportional and Integral (PI) observers as an extension of Luenberger's observer [1] have attracted considerable attention over the past 30 years. By using the integral information as an additional degree of freedom, PI observers can significantly improve the estimation error dynamics. Thus, the design and application of PI observer remain an interesting research topic.

The PI observer was first proposed by [2] for single input single output linear time-invariant system. After that, PI observers for linear multivariable time-varying system [3], linear systems with unknown input distur-

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bances [4], linear uncertain system [5], descriptor system [6] [7], respectively, appeared. However, all of PI observers were designed by proportional to a linear integral of the output estimated error. Presently, P_NI observers have not been developed.

The observers for nonlinear uncertain systems mainly focus on particular classes of nonlinear systems. For the class of Lipschitz nonlinear system, an observer [8] was proposed to permit the simultaneous estimation of the states and the unknown inputs. For a class of nonlinear systems with unknown inputs, the work of [9] has proposed a high gain observer to estimate the states and unknown inputs. In [10], a robust unknown input observer for linear and nonlinear systems was proposed in LMI formulation. An unknown input observer for a class of nonlinear uncertain system, especially consider the linear uncertainties on output matrix, was proposed by [11] and the observer design problem was solved via strong conditions. Most of the observers above were only used to deal with the system, which measured output is a linear combination of the system state, but for the nonlinear case, there are only a few results [4] [5] [7] [9] to deal with the measurement disturbances. Specially, in [9], PI observer design was discussed for a system with constant measurement disturbances. An interesting reformulation of PI observer was given by [5], but it is only valid for attenuating a certain bounded noise in a single output linear uncertain system. A proportional multiple integral observer for descriptor system was proposed by [7], which allows us to decouple or attenuate measurement output disturbances. However, to the best of our knowledge, for the measured output with uncertain nonlinearities on the system states, there are no results.

Therefore, in consideration of the recent progress in the integral control domain, the development of integral observer is so far behind. This point is easy to be seen in the literatures [12]-[18]. General integral control designs based on linear system theory, sliding mode technique and feedback linearization technique were presented by [12]-[14], respectively. In references [15] and [16], general concave and convex integral control along with the bounded integral control actions were proposed, respectively. The method to construct general bounded integral control was presented by [17]. The generalization of integrator and integral control action appeared in [18]. All these nonlinear integral control strategies above stimulate us to develop the nonlinear integral observer and use it to deal with the system with uncertain nonlinearities that appear on both the system dynamics and measured outputs.

Motivated by the cognitions above, this paper proposes a conventional P_NI observer and two kinds of added-order P_NI observers along with their design method, respectively. The main contributions are as follows: 1) A kind of fire new nonlinear integrator and integral action is proposed; 2) The gap that there is not nonlinear integral observer is filled by presenting three kinds of nonlinear integral observers; 3) For the system with uncertain nonlinearities that appear on both the system dynamics and measured outputs, two solutions, that is, mixed and pure added-order P_NI observers, are provided; 4) By linear system theory and Lyapunov method, the conditions on the observer gains to ensure the estimated error to be ultimate boundness, which shrinks to zero as the states and control inputs converge to the equilibrium point, are provided. This means that if the observed system is asymptotically stable, the estimated error dynamics is asymptotically stable, too. Moreover, the highlight point of this paper is that the design of nonlinear integral observer is achieved by linear system theory.

Throughout this paper, we use the notation $\lambda_m(A)$ and $\lambda_M(A)$ to indicate the smallest and largest eigenvalues, respectively, of a symmetric positive define bounded matrix $A(x)$, for any $x \in R^n$. The norm of vector x is defined as $\|x\| = \sqrt{x^{\mathrm{T}} x}$, and that of matrix A is defined as the corresponding induced norm $\|A\| = \sqrt{\lambda_M\left(A^{\mathrm{T}} A\right)}$.

The remainder of the paper is organized as follows: Section 2 describes the system under consideration, assumption and definition. Section 3 addresses the design of nonlinear integral observers. Simulations are provided in Section 4. Conclusions are presented in Section 5.

2. Problem Formulation

Consider the following observable nonlinear system,

$$\begin{cases} \dot{x} = Ax + f(x,u,w) \\ y = Cx + h(x,w) \end{cases} \tag{1}$$

where $x \in D_x \subset R^n$ is the state, $u \in D_u \subset R^p$ is the control input, $y \in D_y \subset R^p$ is the measured output,

$w \in R^l$ is a vector of unknown constant parameters and disturbances, A and C are all constant matrices.

For convenience, we state all definitions, assumptions and theorems for the case when the equilibrium point is at the origin of R^n, that is, $x = 0$.

***Assumption* 1:** No loss of generality, suppose that the function f and h satisfy the equation,

$$\begin{cases} 0 = f(0, u_0, w) \\ 0 = h(0, w) \end{cases} \tag{2}$$

where u_0 is the steady-state control that is needed to maintain equilibrium at the origin of x.

For the purpose of this paper, it is convenient to introduce the following definition.

***Definition* 1:** $F_\Phi(a_\Phi, b_\Phi, x)$ with $a_\Phi > 0$, $b_\Phi > 0$, and $x \in R^n$ denotes the set of all continuous differential increasing functions [17], $\Phi(x) = \begin{bmatrix} \Phi_1(x_1) & \Phi_2(x_2) & \cdots & \Phi_n(x_n) \end{bmatrix}^{\mathrm{T}}$ such that $\Phi(0) = 0$,

$$|\Phi_i(x_i)| \ge b_\Phi \quad \forall\, x_i \in R : |x_i| > a_\Phi \quad \mathrm{d}\Phi_i(x_i)/\mathrm{d}x_i > 0 \quad \forall\, x_i \in R \ (i = 1, 2, \cdots, n)$$

where $|\bullet|$ stands for the absolute value.

Figure 1 depicts the example curves for one component of the function belonging to the function set F_Φ. For instance, for all $x \in R$, the functions, $\arcsin h(x)$, $\tan h(x)$, ax, $ax + bx^3$ ($a > 0$, $b > 0$), $\sin h(x)$, and so on, all belong to function set F_Φ.

3. Observer Design

This section proposes three kinds of nonlinear integral observers, respectively. First, a conventional P_NI observer is proposed to deal with the system without uncertainties in measured output; Second, a mixed added-order P_NI observer was developed for the system with the uncertain nonlinearities that appear on both the system dynamics and measured outputs; Finally, a pure added-order P_NI observer is provided to simplify the design of mixed added-order P_NI observer.

3.1. Conventional P_NI Observer

For the system (1), a conventional P_NI observer can be designed as follows,

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + f_0(\hat{x}, u, \hat{w}) + K_y(y - \hat{y}) + K_\sigma \Phi(\sigma) \\ \hat{y} = C\hat{x} + h_0(\hat{x}, \hat{w}) \\ \dot{\sigma} = \left(\mathrm{d}\Phi(\sigma)/\mathrm{d}\sigma\right)^{-1}(y - \hat{y}) \end{cases} \tag{3}$$

where $\hat{x}$ is the estimated state; $\hat{w}$ is the prescient constant parameters and/or disturbances; $\hat{y}$ is the estimated output; $\Phi(\bullet)$ belongs to the function set F_Φ; $\dot{\sigma}_i = \left(\mathrm{d}\Phi_i(\sigma_i)/\mathrm{d}\sigma_i\right)^{-1}(y_i - \hat{y}_i)$; K_y and K_σ are all gain matrices; $f_0(\hat{x}, u, \hat{w})$ and $h_0(\hat{x}, \hat{w})$ are the nominal models of $f(x, u, w)$ and $h(x, w)$, respectively. Here, $h_0(\hat{x}, \hat{w})$ needs to satisfy $h_0(0, \hat{w}) = 0$.

Thus, the error dynamics can be obtained by subtracting (3) from (1),

$$\begin{cases} \dot{e} = (A - K_y C)e - K_\sigma\left(\Phi(\sigma) - \Phi(\sigma_0)\right) - K_y \delta_h(x, \hat{x}) + \delta_f(x, \hat{x}, u - u_0) \\ \dot{\Phi}(\sigma) = Ce + \delta_h(x, \hat{x}) \end{cases} \tag{4}$$

where $e = x - \hat{x}$ is the estimation state error,

$$\delta_h(x, \hat{x}) = h(x, w) - h(0, w) - h_0(\hat{x}, \hat{w}) + h_0(0, \hat{w}),$$

$$\delta_f(x, \hat{x}, u - u_0) = f(x, u, w) - f(0, u_0, w) - f_0(\hat{x}, u, \hat{w}) + f_0(0, u_0, \hat{w})$$

and K_σ is chosen to be nonsingular and large enough such that the equation,

$$K_\sigma \Phi(\sigma_0) = f_0(0, u_0, \hat{w}) \tag{5}$$

holds as $\dot{e} = 0$, $e = 0$, $\dot{\hat{x}} = 0$, $\hat{x} = 0$ and $u = u_0$ of the error dynamics (4).

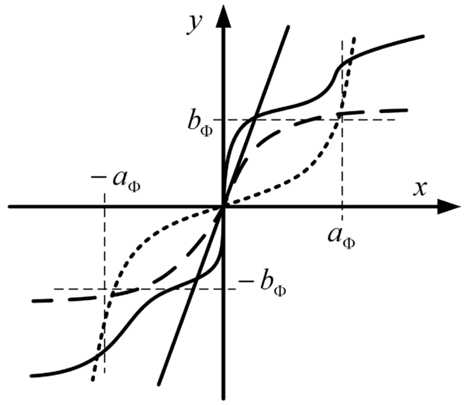

Figure 1. Example curves for one component of the function belonging to the function set F_Φ.

Therefore, we ensure that there is a unique solution σ_0, and then $(0,\sigma_0)$ is the unique equilibrium point of the system (4) in the domain of interest. At the equilibrium point, $\dot{e}=0$, $e=0$, $\dot{\hat{x}}=0$, $\hat{x}=0$, and $u=u_0$, irrespective of the value of u_0.

***Assumption* 2:** By ***Assumption* 1** and the definitions of $\delta_h(x,\hat{x})$ and $\delta_f(x,\hat{x},u-u_0)$, it is reasonable to suppose them to satisfy,

$$\left\|\delta_h(x,\hat{x})\right\| \le \gamma_h^x\|x\|+\gamma_h^e\|e\| \tag{6}$$

$$\left\|\delta_f(x,\hat{x},u-u_0)\right\| \le \gamma_f^x\|x\|+\gamma_f^e\|e\|+\gamma_f^u\|u-u_0\| \tag{7}$$

where γ_h^x, γ_f^x, γ_h^e, γ_f^e and γ_f^u are all positive constants.

Now, the design task is to provide the conditions on the gains K_y and K_σ such that $\lim_{t\to\infty}e(t)=0$. In the absence of $\delta_f(x,\hat{x},u-u_0)$ and $\delta_h(x,\hat{x})$, the asymptotic stability of the error dynamics (4) can be achieved by designing the matrix

$$\Lambda=\begin{bmatrix} A-K_yC & -K_\sigma \\ C & 0 \end{bmatrix}$$

is Hurwitz.

By linear system theory**,** a quadratic Lyapunov function $V(\eta)=\eta^{\mathrm{T}}P\eta$ can be obtained. Where P is the solution of Lyapunov equation $P\Lambda+\Lambda^{\mathrm{T}}P=-Q$ and $\eta=\begin{bmatrix} e & \Phi(\sigma)-\Phi(\sigma_0)\end{bmatrix}^{\mathrm{T}}$.

We use $V(\eta)$ as a Lyapunov function candidate, and then the time derivative of $V(\eta)$ along the trajectories of the error dynamics (4) is,

$$\begin{aligned}\dot{V}(\eta)=\eta^{\mathrm{T}}P\dot{\eta}+\dot{\eta}^{\mathrm{T}}P\eta=\eta^{\mathrm{T}}\left(P\Lambda+\Lambda^{\mathrm{T}}P\right)\eta+\eta^{\mathrm{T}}P\delta\left(K_y,x,\hat{x},u-u_0\right)\\ +\delta^{\mathrm{T}}\left(K_y,x,\hat{x},u-u_0\right)P\eta\end{aligned} \tag{8}$$

where $\delta\left(K_y,x,\hat{x},u-u_0\right)=\begin{bmatrix}\delta_f(x,\hat{x},u-u_0)-K_y\delta_h(x,\hat{x})\\ \delta_h(x,\hat{x})\end{bmatrix}$.

Now, using (2), (6) and (7), we have,

$$\left\|\delta\left(K_y,x,\hat{x},u-u_0\right)\right\|\le\gamma_\delta^e\left(K_y,\gamma_h^e,\gamma_f^e\right)\|e\|+\gamma_\delta^u\left(\gamma_f^u\right)\|u-u_0\|+\gamma_\delta^x\left(K_y,\gamma_h^x,\gamma_f^x\right)\|x\| \tag{9}$$

Substituting (4) into (8), using the inequality (9), Lyapunov equation $P\Lambda+\Lambda^{\mathrm{T}}P=-Q$ with $Q=I$, and $\|e\|\le\|\eta\|$, obtain,

$$\dot{V}(\eta)\le-\rho_\eta^\eta\|\eta\|^2+\rho_\eta^x\|\eta\|\|x\|+\rho_\eta^u\|\eta\|\|u-u_0\| \tag{10}$$

where $\rho_\eta^u = 2\gamma_\delta^u\left(\gamma_f^u\right)\|P\|$, $\rho_\eta^x = 2\gamma_\delta^x\left(K_y,\gamma_h^x,\gamma_f^x\right)\|P\|$ and $\rho_\eta^\eta = 1-2\gamma_\delta^e\left(K_y,\gamma_h^e,\gamma_f^e\right)\|P\|$.

The first term in the right-hand side of the inequality (10) is negative define when,

$$1-2\gamma_\delta^x\left(K_y,\gamma_h^e,\gamma_f^e\right)\|P\|>0 \tag{11}$$

Furthermore, if the following inequality,

$$\|\eta\| \ge \frac{\rho_\eta^x\|x\|+\rho_\eta^u\|u-u_0\|}{\theta\rho_\eta^\eta} \tag{12}$$

holds, it can be verified,

$$\dot{V}(\eta) \le -\left(1-\theta\rho_\eta^\eta\right)\|\eta\|^2 \tag{13}$$

where $0<\theta<1$. Thus, the trajectory of the error dynamics (4) reaches the set

$$\Omega = \left\{\|\eta\| \le \frac{\rho_\eta^x\|x\|+\rho_\eta^u\|u-u_0\|}{\theta\rho_\eta^\eta}\right\}$$

in finite time. The above argument shows that the error dynamics (4) is ultimate boundness with an ultimate bound that decreases as $\|x\|$ and $\|u-u_0\|$ reduce, that is, $\|\eta\|\to 0$ as $\|x\|\to 0$ and $\|u-u_0\|\to 0$. In fact, $\|\eta\|\to 0$ means $\hat{x}\to x$ and $\sigma\to\sigma_0$. This demonstrates that if the nonlinear system (1) is asymptotically stable, and then the error dynamics (4) is asymptotically stable, too. Moreover, a method for estimating the error boundness is provided here. This established the following theorem.

***Theorem* 1:** Under ***Assumption* 1** and **2**, if there exist the gain matrices K_y and K_σ such that the following inequality,

$$\lambda_m\left(K_\sigma\Phi\left(a_\Phi\right)\right) > \left\|f_0\left(0,u_0,\hat{w}\right)\right\| \tag{14}$$

and the inequality (11) hold, and then the error dynamics (4) is ultimate boundness with an ultimate bound that decreases as $\|x\|$ and $\|u-u_0\|$ reduce, that is, $\hat{x}\to x$ and $\sigma\to\sigma_0$ as $x\to 0$ and $u\to u_0$.

***Discussion* 1:** From the error dynamics (4), it is obvious that the observer (3) is only effective for the system with $\delta_h\left(x,\hat{x}\right)=0$, that is, by increasing gain K_y, the uncertain nonlinear action $\delta_f\left(x,\hat{x},u-u_0\right)$ can be effectively attenuated. However, when $\delta_h\left(x,\hat{x}\right)\ne 0$, it results in a dilemma, that is, as follows: 1) By increasing the only unrestricted gain matrix K_σ, the stability of the error dynamics (4) could not be ensured though it can increase the stability margin; 2) The uncertain nonlinear term $\delta_h\left(x,\hat{x}\right)$ in the error dynamics (4), will be amplified unavoidably if the gain K_y is high. Thus, the design of the conventional P_NI observer is not a trivial task because the only way to solve this dilemma is that the precision of model $h_0(\bullet)$ has to be improved. This leads to another worse trouble. Therefore, a new observer is proposed to solve this trouble in the next subsection.

3.2. Mixed Added-Order P_ NI Observer

For making up the shortage of conventional P_NI observer and designing an added-order P_NI observer, the system (1) needs to be added order, which is motivated by the design idea presented by [5], as follows,

$$\begin{cases}\dot{x} = Ax+f\left(x,u,w\right)\\ \dot{x}_0 = y = Cx+h\left(x,w\right)\end{cases} \tag{15}$$

By the augmented system (15), a mixed added-order P_NI observer can be given as,

$$\begin{cases}\dot{\hat{x}} = A\hat{x}+f_0\left(\hat{x},u,\hat{w}\right)+K_y\left(y-\hat{y}\right)+K_x\left(x_0-\hat{x}_0\right)+K_\sigma\Phi\left(\sigma\right)\\ \dot{\hat{x}}_0 = \hat{y} = C\hat{x}+h_0\left(\hat{x},\hat{w}\right)+H_x\left(x_0-\hat{x}_0\right)\\ \dot{\sigma} = \left(\mathrm{d}\Phi\left(\sigma\right)/\mathrm{d}\sigma\right)^{-1}\left(K_y^\sigma\left(y-\hat{y}\right)+K_x^\sigma\left(x_0-\hat{x}_0\right)\right)\end{cases} \tag{16}$$

where $\dot{\sigma}_i = \left(\mathrm{d}\Phi_i\left(\sigma_i\right)/\mathrm{d}\sigma_i\right)^{-1}\left(k_{yi}^\sigma\left(y_i-\hat{y}_i\right)+k_{xi}^\sigma\left(x_{0i}-\hat{x}_{0i}\right)\right)$, K_x, K_y^σ, K_x^σ and H_x are all the gain matrices and the other symbols are the same as these defined in (3).

By the same way as **Subsection 3.1**, the error dynamics can be obtained by subtracting (16) from (15),

$$\begin{cases} \dot{e} = \left(A - K_y C\right)e - \left(K_x - K_y H_x\right)e_0 - K_\sigma\left(\Phi(\sigma) - \Phi(\sigma_0)\right) \\ \qquad - K_y \delta_h\left(x, \hat{x}\right) + \delta_f\left(x, \hat{x}, u - u_0\right) \\ \dot{e}_0 = Ce - H_x e_0 + \delta_h\left(x, \hat{x}\right) \\ \dot{\Phi}(\sigma) = K_y^\sigma Ce + \left(K_x^\sigma - K_y^\sigma H_x\right)e_0 + K_y^\sigma \delta_h\left(x, \hat{x}\right) \end{cases} \tag{17}$$

where $e_0 = x_0 - \hat{x}_0$.

Now, the design task is to provide the conditions on the gain matrices K_x, K_y, H_x, K_σ, K_y^σ and K_x^σ such that $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty} e_0(t) = 0$. In the absence of the uncertain terms $\delta_f\left(x, \hat{x}, u - u_0\right)$ and $\delta_h\left(x, \hat{x}\right)$, the asymptotic stability of the error dynamics (17) can be achieved by designing the matrix,

$$\Lambda_a = \begin{bmatrix} A - K_y C & -\left(K_x - K_y H_x\right) & -K_\sigma \\ C & -H_x & 0 \\ K_y^\sigma C & K_x^\sigma - K_y^\sigma H_x & 0 \end{bmatrix}$$

is Hurwitz.

Now, using (2), (6) and (7), we have,

$$\begin{aligned} \left\|\delta\left(K_y, K_y^\sigma, x, \hat{x}, u - u_0\right)\right\| \le \gamma_\delta^e\left(K_y, K_y^\sigma, \gamma_h^e, \gamma_f^e\right)\|e\| + \gamma_\delta^u\left(\gamma_f^u\right)\|u - u_0\| \\ + \gamma_\delta^x\left(K_y, K_y^\sigma, \gamma_h^x, \gamma_f^x\right)\|x\| \end{aligned} \tag{18}$$

where

$$\delta\left(K_y, K_y^\sigma, x, \hat{x}, u - u_0\right) = \begin{bmatrix} \delta_f\left(x, \hat{x}, u - u_0\right) - K_y \delta_h\left(x, \hat{x}\right) \\ \delta_h\left(x, \hat{x}\right) \\ K_y^\sigma \delta_h\left(x, \hat{x}\right) \end{bmatrix}$$

By the same way as **Subsection 3.1**, we can obtain a quadratic Lyapunov function $V_a\left(\eta_a\right) = \eta_a^{\mathrm{T}} P_a \eta_a$, and then using (18), if the following inequality,

$$1 - 2\gamma_\delta^e\left(K_y, K_y^\sigma, \gamma_h^e, \gamma_f^e\right)\left\|P_a\right\| > 0 \tag{19}$$

holds, we have,

$$\left\|\eta_a\right\| \ge \frac{\rho_\eta^x\|x\| + \rho_\eta^u\|u - u_0\|}{\theta\rho_\eta^\eta} \tag{20}$$

and then the time derivative of $V_a\left(\eta_a\right)$ along the trajectories of the error dynamics (17) satisfies,

$$\dot{V}_a\left(\eta_a\right) \le -\left(1 - \theta\rho_\eta^\eta\right)\left\|\eta_a\right\|^2 \tag{21}$$

where $0 < \theta < 1$, $\eta_a = \left[e^T \quad e_0^T \quad \Phi(\sigma) - \Phi(\sigma_0)\right]^{\mathrm{T}}$,

$$\rho_\eta^u = 2\gamma_\delta^u\left(\gamma_f^u\right)\left\|P_a\right\|, \quad \rho_\eta^x = 2\gamma_\delta^x\left(K_y, K_y^\sigma, \gamma_h^x, \gamma_f^x\right)\left\|P_a\right\|,$$

$$\rho_\eta^\eta = 1 - 2\gamma_\delta^e\left(K_y, K_y^\sigma, \gamma_h^e, \gamma_f^e\right)\left\|P_a\right\|$$

and P_a is the solution of $P_a \Lambda_a + \Lambda_a^{\mathrm{T}} P_a = -I$.

Thus, the trajectory of the error dynamics (17) reaches the set,

$$\Omega_a = \left\{\left\|\eta_a\right\| \le \frac{\rho_\eta^x\|x\| + \rho_\eta^u\|u - u_0\|}{\theta\rho_\eta^\eta}\right\}$$

in finite time. As shown in **Subsection 3.1**, the following theorem can be established.

***Theorem* 2:** Under ***Assumption* 1** and **2**, if there exist the gain matrices K_x, K_y, H_x, K_σ, K_y^σ and K_x^σ such that the following inequality,

$$\lambda_m\left(K_\sigma\Phi\left(a_\Phi\right)\right) > \left\| f_0\left(0, u_0, \hat{w}\right)\right\| \tag{22}$$

and the inequality (19) hold, and then the error dynamics (17) is ultimate boundness with an ultimate bound that decreases as $\|x\|$ and $\|u-u_0\|$ reduce, that is, $\hat{x}_0 \to x_0$, $\hat{x} \to x$ and $\sigma \to \sigma_0$ as $x \to 0$ and $u \to u_0$.

***Remark* 1:** It is obvious that the order of the system (15) and observer (16) are all added. This is why our observer is called the added-order observer. In addition, the observer (16) is designed by using the estimated errors $y-\hat{y}$ and $x_0-\hat{x}_0$, that is the reason why the observer (16) is called mixed added-order observer.

***Discussion* 2:** From the error dynamics (17), it is easy to see that: 1) By increasing K_x and K_x^σ, we can attenuate the actions $\delta_f\left(x, \hat{x}, u-u_0\right)$ and $\delta_h\left(x, \hat{x}\right)$ on $\dot{e}$ and $\dot{\Phi}(\sigma)$, respectively; 2) By increasing H_x, one is that the action $\delta_h\left(x, \hat{x}\right)$ on $\dot{e}_0$ can be limited, another is that it counteracts the actions K_x on $\dot{e}$ and K_x^σ on $\dot{\Phi}(\sigma)$, too; 3) Although K_y and K_y^σ have all the actions to stabilize the error dynamics, they introduce $\delta_h\left(x, \hat{x}\right)$ into the dynamics of $\dot{e}$ and $\dot{\Phi}(\sigma)$, too. Thus, if K_y and K_y^σ are moderate, H_x is large enough, and K_x and K_x^σ are chosen such that $K_x \gg K_y H_x$ and $K_x^\sigma \gg K_y^\sigma H_x$, then we can effectively attenuate the nonlinear actions $\delta_f\left(x, \hat{x}, u-u_0\right)$ and $\delta_h\left(x, \hat{x}\right)$ on the error dynamics. Consequently, we can ensure that the error is bounded.

Obviously, the design method above is too complicated such that some sort of compromise is needed in practice. Therefore, a simplified observer will be proposed in the next subsection.

3.3. Pure Added-Order P_ NI Observer

Based on ***Discussion* 2**, it is obvious that only the actions of K_y and K_y^σ have two kinds of complexion that is positive and negative. Therefore, if we remove them from the observer (16), a pure added-order nonlinear integral observer can be obtained as follows,

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + f_0\left(\hat{x}, u, \hat{w}\right) + K_x\left(x_0 - \hat{x}_0\right) + K_\sigma\Phi(\sigma) \\ \dot{\hat{x}}_0 = \hat{y} = C\hat{x} + h_0\left(\hat{x}, \hat{w}\right) + H_x\left(x_0 - \hat{x}_0\right) \\ \dot{\sigma} = \left(\mathrm{d}\Phi(\sigma)/\mathrm{d}\sigma\right)^{-1}\left(x_0 - \hat{x}_0\right) \end{cases} \tag{23}$$

where $\dot{\sigma}_i = \left(\mathrm{d}\Phi_i\left(\sigma_i\right)/\mathrm{d}\sigma_i\right)^{-1}\left(x_{0i} - \hat{x}_{0i}\right)$, K_x, K_σ and H_x are all the gain matrices and the other symbols are the same as these defined in (3).

By the same way as **Subsection 3.2**, the error dynamics can be obtained by subtracting (23) from (15),

$$\begin{cases} \dot{e} = Ae - K_x e_0 - K_\sigma\left(\Phi(\sigma) - \Phi\left(\sigma_0\right)\right) + \delta_f\left(x, \hat{x}, u-u_0\right) \\ \dot{e}_0 = Ce - H_x e_0 + \delta_h\left(x, \hat{x}\right) \\ \dot{\Phi}(\sigma) = e_0 \end{cases} \tag{24}$$

and then by letting $K_y = K_y^\sigma = 0$ in **Subsection 3.2**, a simpler theorem can be established.

***Theorem* 3:** Under ***Assumption* 1** and **2**, if there exist the gain matrices K_x, H_x and K_σ such that the following inequalities,

$$\lambda_m\left(K_\sigma\Phi\left(a_\Phi\right)\right) > \left\| f_0\left(0, u_0, \hat{w}\right)\right\| \tag{25}$$

$$1 - 2\gamma_\delta^e\left(\gamma_h^e, \gamma_f^e\right)\left\|P_a\right\| > 0 \tag{26}$$

hold, and then the error dynamics (24) is ultimate boundness with an ultimate bound that decreases as $\|x\|$ and $\|u-u_0\|$ reduce, that is, $\hat{x}_0 \to x_0$, $\hat{x} \to x$ and $\sigma \to \sigma_0$ as $x \to 0$ and $u \to u_0$.

***Remark* 2:** It is easy to see that the observer (23) is designed only by the estimated error $x_0 - \hat{x}_0$, that is the reason why the observer (23) is called pure added-order P_NI observer.

***Discussion* 3:** From the error dynamics (24) and demonstration above, it is obvious that: 1) K_x, K_σ and

H_x are all independent gain matrices; 2) The matrix Λ_a and the bound of the nonlinear actions (18) can all be simplified; 3) By increasing K_x and H_x, we can attenuate the actions $\delta_f\left(x,\hat{x},u-u_0\right)$ and $\delta_h\left(x,\hat{x}\right)$ on $\dot{e}$ and $\dot{\Phi}\left(\sigma\right)$, respectively. Thus, the observer (23) not only can effectively deal with the uncertain nonlinear actions on system (1) but also the stability of the error dynamics is easier to be achieved. Moreover, since the integral action can attenuate measurement noise, the observer (23) can be suitable for handling measurement noise, too.

***Discussion* 4:** Although the works of [5] [7] and the observers (15) and (23) all use the same reformulation proposed by [5], their main differences are as follows: 1) The integral action and integrator, here they are all nonlinear, but they are all linear in [5] [7]; 2) The observed system, here it is used to deal with the uncertain nonlinear system, however, it is used to deal with the uncertain linear system and descriptor system in [5] [7], respectively; 3) The twice integrals of measured output and estimated one, they are used to decouple or attenuate measurement output disturbances in [7], but this paper uses it to counteract the unknown constant uncertainties produced by the model errors, prescient constant parameters and unknown steady-state control input.

***Discussion* 5:** Compared with the integrators and integral actions proposed by [2]-[18], the main differences are that: 1) The integrator and integral action: here are all nonlinear; however, they are all linear in [2]-[11], except for the reference [11], where the diffeomorphism is used as the integrator; 2) The indispensable components to construct the integrator: here are the linear form on the estimated errors; however, they are taken as the partial derivative of Lyapunov function in [14]-[16] and a general function on all the system states in [17] [18], respecttively; 3) The integral actions: here not only include bounded integral actions, such as $\Phi\left(\sigma\right)=\tanh\left(\sigma\right)$, but also contains the unbounded one; however, they are all bounded in [15]-[17]; 4) The correlations between the integrator and integral action: here they are closely related; however, they are independent of each other in [2]-[11] and [18]. Therefore, the nonlinear integrator and integral action proposed here are fire new.

***Remark* 3:** From the stability analysis of **Subsections 3.1 - 3.3**, it is obvious that: Just the integrator is taken as the product of estimated error and reciprocal of derivative $\mathrm{d}\Phi\left(\sigma\right)/\mathrm{d}\sigma$, the time derivative of integral action can be transformed into the linear form on the estimated error. Just with this ingenious mathematical transformation [15], we can use linear system theory to analyze the stability of the error dynamics with the nonlinear integral action and integrator. As a result, this is a highlight point of this paper.

4. Simulations

Consider the pendulum system [19] described by,

$$\ddot{\theta}=-a\sin\theta-b\dot{\theta}+cT$$

where $a=g/l>0$, $b=k/m>0$, $c=1/ml^2>0$, θ is the angle subtended by the rod and the vertical axis, and T is the torque applied to the pendulum. View T as the control input and suppose we want to regulate θ to r. For the purpose of this paper, we add an uncertain nonlinear action $0.5x_1+\sin\left(3x_1+2x_2\right)$ on the measured output y, and then taking $x_1=\theta-r$, $x_2=\dot{\theta}$ and $u=T$, the pendulum system with the extra nonlinear action can be written as,

$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=-a\sin\left(x_1+r\right)-bx_2+cu\\ y=1.5x_1+\sin\left(3x_1+2x_2\right)\end{cases}$$

By the design method proposed here, the augmented system can be given as,

$$\begin{cases}\dot{x}_0=1.5x_1+\sin\left(3x_1+2x_2\right)\\ \dot{x}_1=x_2\\ \dot{x}_2=-a\sin\left(x_1+r\right)-bx_2+cu\end{cases}$$

and then, the pure added-order P_NI observer can be given as,

$$\begin{cases} \dot{\hat{x}}_0 = \hat{x}_1 + h_0\left(x_0 - \hat{x}_0\right) \\ \dot{\hat{x}}_1 = \hat{x}_2 + k_1\left(x_0 - \hat{x}_0\right) \\ \dot{\hat{x}}_2 = -a\sin\left(\hat{x}_1 + r\right) + cu + k_2\left(x_0 - \hat{x}_0\right) + k_\sigma \sinh\left(\hat{\sigma}\right) \\ \dot{\hat{\sigma}} = \cosh^{-1}\left(\hat{\sigma}\right)\left(x_0 - \hat{x}_0\right) \end{cases}$$

By the design method proposed here, we can take $h_0 = 100$, $k_1 = 150$, $k_2 = 1500$ and $k_\sigma = 1500$ such that the matrix

$$\Lambda = \begin{bmatrix} -h_0 & 1 & 0 & 0 \\ -k_1 & 0 & 1 & 0 \\ -k_2 & 0 & 0 & -k_\sigma \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is Hurwitz.

Therefore, the control input can be taken as,

$$\begin{cases} u = -8\hat{x}_1 - 6\hat{x}_2 - 5\sigma \\ \dot{\sigma} = 8\hat{x}_1 + 5\hat{x}_2 \end{cases}$$

For demonstrating the performance of the pure added-order observer, the simulations are implemented under normal and perturbed parameter cases, respectively.

Normal case: The initial states are $x_0 = x_2 = 0$, $x_1 = -2$, $\hat{x}_1 = -2$ and $\hat{x}_0 = \hat{x}_2 = 0$; the system parameters are $a = c = 10$ and $b = 1$.

Perturbed case: The initial states are $x_1 = -2$, $x_0 = 0$, $x_2 = -1$, $\hat{x}_1 = -2$ and $\hat{x}_0 = \hat{x}_2 = 0$; the system parameters are $a = 10$, $b = 0.5$ and $c = 5$, corresponding to doubling of the mass.

Figure 2 and **Figure 3** showed the simulation results under the normal (solid line) and perturbed (dashed line) cases. As shown on **Figure 2**, the good control performance is still preserved, even under the perturbed case. This demonstrated that the observer (23) can be applied to the observer-based control. **Figure 3** clearly shows that the estimated velocity error quickly shrinks to zero as the position tends to the equilibrium point. This not

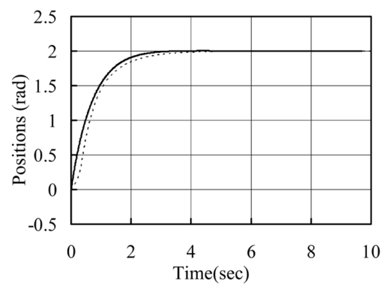

Figure 2. System output under the normal (solid line) and perturbed (dashed line) cases.

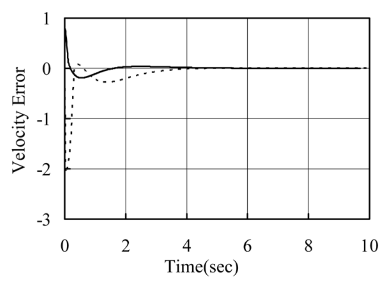

Figure 3. Velocity error under the normal (solid line) and perturbed (dashed line) cases.

only verified the justification of ***Theorem* 3** but also shows that the observer (23) has strong robustness and can effectively deal with the uncertain nonlinearities on both the system dynamics and measured outputs.

5. Conclusion

This paper proposed a conventional P_NI observer and two kinds of added-order P_NI observers along with their design method. The main contributions are as follows: 1) A kind of fire new nonlinear integrator and integral action is proposed; 2) The gap that there is not nonlinear integral observer is filled by presenting three kinds of nonlinear integral observers; 3) For the system with uncertain nonlinearities that appear on both the system dynamics and measured outputs, two solutions, that is, mixed and pure added-order P_NI observers, are provided; 4) The conditions on the observer gains to ensure the estimated error to be ultimate boundness, which shrinks to zero as the states and control inputs converge to the equilibrium point, are provided. This means that if the observed system is asymptotically stable, the estimated error dynamics is asymptotically stable, too. In addition, the highlight point of this paper is that the design of nonlinear integral observer was achieved by linear system theory.

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