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# Generalizations of a Matrix Inequality

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## ABSTRACT

**In this paper, some new generalizations of the matrix form of the Brunn-Minkowski inequality are presented.**

## KEYWORDS

**Brunn-Minkowski Inequality; Positive Definite Matrix; Determinant Differences**

## *1. Introduction*

The well-known Brunn-Minkowski inequality is one of the most important inequalities in geometry. There are many other interesting results related to the Brunn-Minkowski inequality (see [1-8]). The matrix form of the Brunn-Minkowski inequality (see [9,10]) asserts that if $A$ and $B$ are two positive definite matrices of order $n$ and $0<\lambda<1$, then

$$\left|\lambda A+(1-\lambda)B\right|^{\frac{1}{n}}\geq\lambda\left|A\right|^{\frac{1}{n}}+(1-\lambda)\left|B\right|^{\frac{1}{n}}, \tag{1}$$

with equality if and only if $A=cB\,(c\geq 0)$, where $|A|$ denotes the determinant of $A$.

Let $\mathbb{R}^{n\times n}$ denote the set of $n\times n$ real symmetry matrices. Let $I_n$ denote $n\times n$ unit matrix. We use the notation $A>0\,(A\geq 0)$ if $A$ is a positive definite (positive semi-definite) matrix, and $A^*$ denotes the transpose of $A$. Let $A,B\in\mathbb{R}^{n\times n}$, then $A>B\,(A\geq B)$ if and only if $A-B>0\,(A-B\geq 0)$.

If $A\in\mathbb{R}^{n\times n}$, then there exists a unitary matrix $U$ such as

$$A=U^*[\lambda_1,\cdots,\lambda_n]U,$$

where $[\lambda_1,\cdots,\lambda_n]$ is a diagonal matrix $(\lambda_i\delta_{ij})$, and $\lambda_1,\cdots,\lambda_n$ are the eigenvalues of $A$, each appearing as its multiplicity. Assume now that $f(\lambda_i)\in\mathbb{R}$ is well defined. Then $f(A)$ may be defined by (see e.g. [11, p. 71] or [12, p. 90])

$$f(A)=U^*\left[f(\lambda_1),\cdots,f(\lambda_n)\right]U. \tag{2}$$

In this paper, some new generalizations of the matrix form of the Brunn-Minkowski inequality are presented. One of our main results is the following theorem.

**Theorem 1.1.** *Let $A$, $B$ be positive definite commuting matrix of order $n$ with eigenvalues in the interval $I$. If $f$ is a positive concave function on $I$ and $0<\lambda<1$, then*

$$\left|f(\lambda A+(1-\lambda)B)\right|^{\frac{1}{n}}\geq\lambda\left|f(A)\right|^{\frac{1}{n}}+(1-\lambda)\left|f(B)\right|^{\frac{1}{n}} \tag{3}$$

with equality if and only if $f$ is linear and $f(A)=cf(B)(c\ge 0)$.

Let $A,B\in\mathbb{R}^{n\times n}$, if $A\ge B$. We can define the *determinant differences function* of $A$ and $B$ by

$$D_d(A,B)=|A|-|B|.$$

The following theorem gives another generalization of (1).

**Theorem 1.2.** *Let* $A$, $B$ *be positive definite commuting matrix of order* $n$ *with eigenvalues in the interval* $I$ *and* $1\in I$. *Let* $f$ *be a positive function on* $I$ *and a and b be two nonnegative real numbers such that*

$$f(A)>af(I_n),\ f(B)>bf(I_n).$$

Then

$$D_d\left(f(A)+f(B),(a+b)f(I_n)\right)^{\frac{1}{n}}\ge D_d\left(f(A),af(I_n)\right)^{\frac{1}{n}}+D_d\left(f(B),bf(I_n)\right)^{\frac{1}{n}} \tag{4}$$

with equality if and only if $a^{-1}f(A)=b^{-1}f(B)$.

*Remark* 1. Let $f(t)=t$ in Theorem 1.1 or let $f(t)=t$ and $a=b=0$ in Theorem 1.2. We can both obtain (1). Hence Theorem 1.1 and Theorem 1.2 are generalizations of (1).

## 2. Proofs of Theorems

To prove the theorems, we need the following lemmas:

**Lemma 2.1.** ([13], p.472) *Let* $A,B\in\mathbb{R}^{n\times n}$, $A>B>0$. *Then*

$$|A|>|B|.$$

**Lemma 2.2.** ([13], p.50) *Let* $A,B\in\mathbb{R}^{n\times n}$, $A>0$, $B>0$. *If* $A$ *and* $B$ *are commute, then exists a unitary matrix* $U$ *such that*

$$U^*AU=[a_1,a_2,\cdots,a_n]\ \text{ and }\ U^*BU=[b_1,b_2,\cdots,b_n].$$

**Lemma 2.3.** ([14], p.35) *Let* $x_i\ge 0,\ y_i\ge 0\ (i=1,2,\cdots,n)$. *Then*

$$\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}+\left(\prod_{i=1}^n y_i\right)^{\frac{1}{n}}\le\left(\prod_{i=1}^n (x_i+y_i)\right)^{\frac{1}{n}},$$

*with equality if and only if* $x_i=\nu y_i$, *where* $\nu$ *is a constant.*

This is a special case of Maclaurin's inequality.

*Proof of Theorem* 1.1.

Since $A$ and $B$ are commuted, by lemma 2.2, there exists a unitary matrix $U$ such that

$$A=U^*[a_1,a_2,\cdots,a_n]U\ \text{ and }\ B=U^*[b_1,b_2,\cdots,b_n]U.$$

Hence,

$$\lambda A+(1-\lambda)B=U^*\left[\lambda a_1+(1-\lambda)b_1,\lambda a_2+(1-\lambda)b_2,\cdots,\lambda a_n+(1-\lambda)b_n\right]U.$$

By (2), we have

$$f(A)=U^*\left[f(a_1),f(a_2),\cdots,f(a_n)\right]U,$$

$$f(B)=U^*\left[f(b_1),f(b_2),\cdots,f(b_n)\right]U,$$

and

$$f(\lambda A+(1-\lambda)B)=U^*\left[f(\lambda a_1+(1-\lambda)b_1),f(\lambda a_2+(1-\lambda)b_2),f(\lambda a_n+(1-\lambda)b_n)\right]U.$$

Since $f$ is a concave function, by lemma 2.3, we get

$$\left|f\left(\lambda A+(1-\lambda)B\right)\right|^{\frac{1}{n}}=\left(\prod_{i=1}^{n}f\left(\lambda a_i+(1-\lambda)b_i\right)\right)^{\frac{1}{n}}$$

$$\geq\left(\prod_{i=1}^{n}\left[\lambda f(a_i)+(1-\lambda)f(b_i)\right]\right)^{\frac{1}{n}} \tag{5}$$

$$\geq\lambda\left(\prod_{i=1}^{n}f(a_i)\right)^{\frac{1}{n}}+(1-\lambda)\left(\prod_{i=1}^{n}f(b_i)\right)^{\frac{1}{n}} \tag{6}$$

$$=\lambda\left|f(A)\right|^{\frac{1}{n}}+(1-\lambda)\left|f(B)\right|^{\frac{1}{n}}.$$

Now we consider the conditions of equality holds. Since $f$ is a concave function, the equality of (5) holds if and only if $f$ is linear. By the equality of Lemma 2.3, the equality of (6) holds if and only if $f(a_i)=cf(b_i)$, which means $f(A)=cf(B)$. So the equality of (3) holds if and only if $f$ is linear and $f(A)=cf(B)\,(c\geq 0)$. This completes the proof of the Theorem 1.1.

Applying the arithmetic-geometric mean inequality to the right side of (3), we get the following corollary.

**Corollary 2.4.** *Let $A$, $B$ be positive definite commuting matrix of order $n$ with eigenvalues in the interval $I$. If $f$ is a positive concave function on $I$ and $0<\lambda<1$, then*

$$\left|f\left(\lambda A+(1-\lambda)B\right)\right|\geq\left|f(A)\right|^{\lambda}\left|f(B)\right|^{1-\lambda},$$

with equality if and only if $A=B$.

Taking for $f(t)=t$ in Corollary 2.4, we obtain the Fan Ky concave theorem.

*Proof of Theorem* 1.2.

As in the proof of Theorem 1.1, since $A$ and $B$ are commuted, by lemma 2.2, there exists a unitary matrix $U$ such that

$$f(A)=U^*\left[f(a_1),f(a_2),\cdots,f(a_n)\right]U$$

and

$$f(B)=U^*\left[f(b_1),f(b_2),\cdots,f(b_n)\right]U.$$

So

$$\left|f(A)\right|=\prod_{i=1}^{n}f(a_i),\quad \left|f(B)\right|=\prod_{i=1}^{n}f(b_i),$$

$$\left|f(A)+f(B)\right|=\prod_{i=1}^{n}\left(f(a_i)+f(b_i)\right).$$

It is easy to see that (4) holds if and only if

$$\begin{aligned}&\left(\prod_{i=1}^{n}\left(f(a_i)+f(b_i)\right)-\left[(a+b)f(1)\right]^n\right)^{\frac{1}{n}}\\ &\geq\left(\prod_{i=1}^{n}f(a_i)-\left[af(1)\right]^n\right)^{\frac{1}{n}}+\left(\prod_{i=1}^{n}f(b_i)-\left[bf(1)\right]^n\right)^{\frac{1}{n}}.\end{aligned} \tag{7}$$

Since $f(A)>af(I_n)$, $f(B)>bf(I_n)$, by Lemma 2.1, we have

$$\prod_{i=1}^{n}f(a_i)>\left[af(1)\right]^n,\quad \prod_{i=1}^{n}f(b_i)>\left[bf(1)\right]^n.$$

Now we prove (7). Put

$$X^n = \prod_{i=1}^{n} f(a_i) - [af(1)]^n, \quad Y^n = \prod_{i=1}^{n} f(b_i) - [bf(1)]^n.$$

Then

$$X^n + [af(1)]^n = \prod_{i=1}^{n} f(a_i), \quad Y^n + [bf(1)]^n = \prod_{i=1}^{n} f(b_i).$$

Applying Minkowski inequality, we have

$$\left((X+Y)^n + [(a+b)f(1)]^n\right)^{\frac{1}{n}} \le \left(X^n + [af(1)]^n\right)^{\frac{1}{n}} + \left(Y^n + [bf(1)]^n\right)^{\frac{1}{n}}$$
$$= \left(\prod_{i=1}^{n} f(a_i)\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} f(b_i)\right)^{\frac{1}{n}}$$

Using the Lemma 2.3 to the right of the above inequlity, we obtain

$$\left((X+Y)^n + [(a+b)f(1)]^n\right)^{\frac{1}{n}} \le \left(\prod_{i=1}^{n} (f(a_i) + f(b_i))\right)^{\frac{1}{n}},$$

which implies that

$$(X+Y)^n \le \prod_{i=1}^{n} (f(a_i) + f(b_i)) - [(a+b)f(1)]^n.$$

It follows that

$$X + Y \le \left(\prod_{i=1}^{n} (f(a_i) + f(b_i)) - [(a+b)f(1)]^n\right)^{\frac{1}{n}},$$

which is just the inequality (7).

By the equality conditions of Minkowski inequality and Lemma 2.3, the equality (1.4) holds if and only if $a^{-1}f(a_i) = b^{-1}f(b_i)$, which means $a^{-1}f(A) = b^{-1}f(B)$. Thus we complete the proof of Theorem 1.2.

Taking for $f(t) = t$ in Theorem 1.2, we obtain the following corollary.

**Corollary 2.5.** [7] *Let* $A$, $B$ *be positive definite commuting matrix of order* $n$ *and a and b be two nonnegative real numbers such that*

$$A > aI_n, \ B > bI_n.$$

*Then*

$$\left(|A+B| - |(a+b)I_n|\right)^{\frac{1}{n}} \ge \left(|A| - |aI_n|\right)^{\frac{1}{n}} + \left(|B| - |bI_n|\right)^{\frac{1}{n}},$$

*with equality if and only if* $a^{-1}A = b^{-1}B$.

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