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An Optimal Double Inequality among the One-Parameter, Arithmetic and Geometric Means

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ABSTRACT

In the present paper, we answer the question: for $0<\alpha<1$ fixed, what are the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that the double inequality $J_p(a,b)<\alpha A(a,b)+(1-\alpha)G(a,b)<J_q(a,b)$ holds for all $a,b>0$ with $a\neq b$? where for $p\in R$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive real numbers a and b are defined by $J_p(a,b)=\begin{cases} a, & a\neq b, \\ \dfrac{p\left(a^{p+1}-b^{p+1}\right)}{(p+1)\left(a^p-b^p\right)}, & a\neq b, p\neq -1,0, \\ \dfrac{ab(\log a-\log b)}{a-b}, & a\neq b, p=-1, \\ \dfrac{a-b}{\log a-\log b}, & a\neq b, p=0, \end{cases}$

$A(a,b)=\dfrac{a+b}{2}$ and $G(a,b)=\sqrt{ab}$, respectively.

Keywords: Optimal Double Inequality; One-Parameter Mean; Arithmetic Mean; Geometric Mean

1. Introduction

For $p\in R$, the one-parameter mean $J_p(a,b)$, arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$ of two positive real numbers a and b are defined by

$$J_p(a,b)=\begin{cases} a, & a\neq b, \\ \dfrac{p\left(a^{p+1}-b^{p+1}\right)}{(p+1)\left(a^p-b^p\right)}, & a\neq b, p\neq -1,0, \\ \dfrac{ab(\log a-\log b)}{a-b}, & a\neq b, p=-1, \\ \dfrac{a-b}{\log a-\log b}, & a\neq b, p=0, \end{cases} \tag{1}$$

$A(a,b)=\dfrac{a+b}{2}$ and $G(a,b)=\sqrt{ab}$, respectively.

It is well-known that the one-parameter mean is continuous and strictly increasing with respect to $p\in R$ for fixed $a,b>0$ with $a\neq b$. Many means are special cases of the one-parameter mean, for example:

$J_1(a,b)=A(a,b)$ is the arithmetic mean,

$J_{1/2}(a,b)=He(a,b)$ is the Heronian mean,

$J_{-1/2}(a,b)=G(a,b)$ is the geometric mean, and

$J_{-2}(a,b)=H(a,b)$ is the harmonic mean.

The one-parameter mean $J_p(a,b)$ and its inequalities have been studied intensively, see [1-6].

The purpose of this paper is to answer the question: for $0<\alpha<1$, what are the greatest value $p(\alpha)$ and the least value $q(\alpha)$ such that the double inequality

$J_p(a,b)<\alpha A(a,b)+(1-\alpha)G(a,b)<J_q(a,b)$ holds for all $a,b>0$ with $a\neq b$?

2. Main Result

The main result of this paper is the following theorem.

Theorem 2.1. Let $0<\alpha<1$. Then for any $a,b>0$ with $a\neq b$, we have

1) $J_{\frac{3\alpha-1}{2}}(a,b)=\alpha A(a,b)+(1-\alpha)G(a,b)=J_{\frac{\alpha}{2-\alpha}}(a,b)$ for $\alpha=\frac{2}{3}$,

2) $J_{\frac{3\alpha-1}{2}}(a,b)<\alpha A(a,b)+(1-\alpha)G(a,b)<J_{\frac{\alpha}{2-\alpha}}(a,b)$ for $\alpha\in\left(0,\frac{2}{3}\right)$,

3) $J_{\frac{\alpha}{2-\alpha}}(a,b)<\alpha A(a,b)+(1-\alpha)G(a,b)<J_{\frac{3\alpha-1}{2}}(a,b)$ for $\alpha\in\left(\frac{2}{3},1\right)$.

The numbers $\frac{3\alpha-1}{2}$ and $\frac{\alpha}{2-\alpha}$ in 2) and 3) are optimal.

In order to prove Theorem 2.1, we need a preliminary lemma.

Lemma 2.1. For $t>1$, one has

$$g(t)=\frac{t^2-1}{2\log t}-\frac{t^2+4t+1}{6}<0 \tag{2}$$

Proof. Simple calculations lead to

$$g(t)=\frac{t^2+4t+1}{6\log t}g_1(t), \tag{3}$$

$$g_1(t)=\frac{3\left(t^2-1\right)}{t^2+4t+1}-\log t, \tag{4}$$

$$\lim_{t\to1^+}g_1(t)=0, \tag{5}$$

$$g_1'(t)=\frac{-t^4+4t^3-6t^2+4t-1}{t\left(t^2+4t+1\right)^2}=\frac{-(t-1)^4}{t\left(t^2+4t+1\right)^2}<0 \tag{6}$$

(2) follows from (3)-(6).

Proof of Theorem 2.1. Without loss of generality we assume $a>b$ and take $t=\sqrt{a/b}>1$. We first consider the case $\alpha=\frac{2}{3}$. 1) follows from

$$J_{\frac{1}{2}}(t,1)=He(t,1)=\frac{t+\sqrt{t}+1}{3}=\frac{2}{3}A(t,1)+G(t,1).$$

From now on we assume $\alpha\neq\frac{2}{3}$. Let $p\in\left\{\frac{3\alpha-1}{2},\frac{\alpha}{2-\alpha}\right\}$, then (1) leads to

$$\begin{aligned}f(t)&=\left[\alpha A\left(t^2,1\right)+(1-\alpha)G\left(t^2,1\right)\right]-J_p\left(t^2,1\right)\\&=\frac{h(t)}{2(p+1)\left(t^{2p}-1\right)},\end{aligned} \tag{7}$$

where

$$\begin{aligned}h(t)&=(\alpha p-2p+\alpha)t^{2p+2}+2(1-\alpha)(p+1)t^{2p+1}\\&\quad+\alpha(p+1)t^{2p}-\alpha(p+1)t^2\\&\quad-2(1-\alpha)(p+1)t-(\alpha p-2p+\alpha).\end{aligned}$$

Simple calculations lead to

$$\lim_{t\to1^+}h(t)=0, \tag{8}$$

$$\begin{aligned}h'(t)&=2(p+1)(\alpha p-2p+\alpha)t^{2p+1}\\&\quad+2(2p+1)(1-\alpha)(p+1)t^{2p}+2\alpha p(p+1)t^{2p-1}\\&\quad-2\alpha(p+1)t-2(1-\alpha)(p+1)\\&=2(p+1)h_1(t),\end{aligned}$$

where

$$\begin{aligned}h_1(t)&=(\alpha p-2p+\alpha)t^{2p+1}+(1-\alpha)(2p+1)t^{2p}\\&\quad+p\alpha t^{2p-1}-\alpha t-(1-\alpha),\end{aligned}$$

$$\lim_{t\to1^+}h_1(t)=0, \tag{9}$$

$$\begin{aligned}h_1'(t)&=(2p+1)(\alpha p-2p+\alpha)t^{2p}\\&\quad+2p(1-\alpha)(2p+1)t^{2p-1}\\&\quad+p(2p-1)\alpha t^{2p-2}-\alpha\end{aligned}$$

$$\lim_{t\to1^+}h_1'(t)=0, \tag{10}$$

$$\begin{aligned}h_1''(t)&=2p(2p+1)(\alpha p-2p+\alpha)t^{2p-1}\\&\quad+2p(2p-1)(1-\alpha)(2p+1))t^{2p-2}\\&\quad+2(p-1)p(2p-1)\alpha t^{2p-3}\\&=2pt^{2p-3}h_2(t),\end{aligned} \tag{11}$$

where

$$\begin{aligned}h_2(t)&=(2p+1)(\alpha p-2p+\alpha)t^2t\\&\quad+(2p-1)(2p+1)(1-\alpha)\\&\quad+(p-1)(2p-1)\alpha\end{aligned}$$

$$\lim_{t\to1^+}h_2(t)=3\alpha-2p-1, \tag{12}$$

$$\begin{aligned}h_2'(t)&=2(2p+1)(\alpha p-2p+\alpha)t\\&\quad+(2p-1)(2p+1)(1-\alpha)\\&=(2p+1)h_3(t),\end{aligned} \tag{13}$$

where

$$h_3(t) = 2(\alpha p - 2p + \alpha)t + (2p-1)(1-\alpha), \quad (14)$$

$$\lim_{t\to 1^+} h_3(t) = 3\alpha - 2p - 1, \quad (15)$$

$$h_3'(t) = 2(\alpha p - 2p + \alpha). \quad (16)$$

We shall distinguish between two cases.

Case 1. $p = \frac{3\alpha - 1}{2}$. The left-hand side inequality of 2) for $\alpha = \frac{1}{3}$ follows from Lemma 2.1 because in this case

$$J_0(t^2,1) - \left[\frac{1}{3}A(t^2,1) + \frac{2}{3}G(t^2,1)\right] = g(t) < 0$$

for all $t > 1$. In the sequel we assume $\alpha \neq \frac{1}{3}$.

We clearly see from (16) that

$$h_3'(t) = (3\alpha - 2)(\alpha - 1)\begin{cases} < 0, & \alpha \in \left(0, \frac{2}{3}\right), \\ > 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Thus $h_3(t)$ is strictly decreasing for $\alpha \in \left(0, \frac{2}{3}\right)$ and strictly increasing for $\alpha \in \left(\frac{2}{3}, 1\right)$. (2.14) yields $h_3(1^+) = 0$, then $h_3(t) < 0$ for $\alpha \in \left(0, \frac{2}{3}\right)$ and $h_3(t) > 0$ for $\alpha \in \left(\frac{2}{3}, 1\right)$. The same reasoning applies to $h_2'(t)$ and $h_2(t)$ as well, and noticing (13) and (12), one has

$$h_2(t)\begin{cases} > 0, & \alpha \in \left(0, \frac{2}{3}\right), \\ < 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

This result together with (11) implies

$$h_1''(t)\begin{cases} < 0, & \alpha \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right), \\ > 0, & \alpha \in \left(\frac{1}{3}, \frac{2}{3}\right). \end{cases}$$

Thus $h_1'(t)$ is strictly decreasing for $\alpha \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right)$ and strictly increasing for $\alpha \in \left(\frac{1}{3}, \frac{2}{3}\right)$. The same reasoning applies to $h_1'(t), h_1(t)$ and $h(t)$ as well, and applying (8)-(10), we derive

$$h(t)\begin{cases} < 0, & \alpha \in \left(0, \frac{1}{3}\right) \cup \left(\frac{2}{3}, 1\right), \\ > 0, & \alpha \in \left(\frac{1}{3}, \frac{2}{3}\right). \end{cases}$$

Since $t^{2p} - 1 < 0$ for $\alpha \in \left(0, \frac{1}{3}\right)$ and $t^{2p} - 1 > 0$ for $\alpha \in \left(\frac{1}{3}, 1\right)$, then we know from (7) that

$$f(t)\begin{cases} > 0, & \alpha \in \left(0, \frac{2}{3}\right) \\ < 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

This implies the left-hand side of 2) and the right-hand side of 3).

Case 2. $p = \frac{\alpha}{2 - \alpha}$. From (14) we know that

$$h_3(t) = \frac{(3\alpha - 2)(1 - \alpha)}{2 - \alpha}\begin{cases} < 0, & \alpha \in \left(0, \frac{2}{3}\right) \\ > 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

From (13) we know that $h_2'(t) < 0$ for $\alpha \in \left(0, \frac{2}{3}\right)$ and $h_2'(t) > 0$ for $\alpha \in \left(\frac{2}{3}, 1\right)$. This implies $h_2(t)$ is strictly decreasing for $\alpha \in \left(0, \frac{2}{3}\right)$ and strictly increasing for $\alpha \in \left(\frac{2}{3}, 1\right)$. From (12) we know

$$\lim_{t\to 1^+} h_2(t) = \frac{(3\alpha - 2)(1 - \alpha)}{2 - \alpha}\begin{cases} < 0, & \alpha \in \left(0, \frac{2}{3}\right) \\ > 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

Therefore

$$h_2(t)\begin{cases} < 0, & \alpha \in \left(0, \frac{2}{3}\right), \\ > 0, & \alpha \in \left(\frac{2}{3}, 1\right). \end{cases}$$

(11) implies $h_1''(t)$ has the same property as $h_2(t)$, thus $h_1'(t)$ is strictly decreasing for $\alpha \in \left(0, \frac{2}{3}\right)$ and strictly increasing for $\alpha \in \left(\frac{2}{3}, 1\right)$. The same reasoning applies to $h_1(t)$, $h'(t)$ and $h(t)$ as well, and notic-

ing (9) and (8), one has

$$h(t)\begin{cases}<0, & \alpha\in\left(0,\dfrac{2}{3}\right),\\ >0, & \alpha\in\left(\dfrac{2}{3},1\right).\end{cases}$$

which together with (7) implies

$$f(t)\begin{cases}<0, & \alpha\in\left(0,\dfrac{2}{3}\right),\\ >0, & \alpha\in\left(\dfrac{2}{3},1\right).\end{cases}$$

This implies the right-hand side of 2) and the left-hand side of 3).

We are now in the position to prove the constants $\frac{3\alpha-1}{2}$ and $\frac{\alpha}{2-\alpha}$ are optimal.

For any ε (positive or negative, with $|\varepsilon|$ sufficiently small) we consider the case $p=\frac{3\alpha-1}{2}+\varepsilon$. (12) implies

$$\lim_{t\to 1^+} h_2(t)\begin{cases}<0, & \varepsilon>0,\\ >0, & \varepsilon<0.\end{cases}$$

By the continuity of $h_2(t)$, there exists $\delta_1=\delta_1(\varepsilon)>0$ such that

$$h_2(t)\begin{cases}<0, & \text{for } 1<t<1+\delta_1 \text{ and } \varepsilon>0,\\ >0, & \text{for } 1<t<1+\delta_1 \text{ and } \varepsilon<0.\end{cases}$$

By (11), $ph_1''(t)$ as the same property as $h_2(t)$. The same reasoning applies to $ph_1'(t)$, $ph_1(t)$, $ph'(t)$ and $ph(t)$ as well, and noticing (10)-(8), we know $ph(t)$ has the same property as $h_2(t)$. By (7) one has

$$f(t)\begin{cases}<0, & \varepsilon>0,\\ >0, & \varepsilon<0.\end{cases}$$

This proves the optimality for $\frac{3\alpha-1}{2}$.

To prove the optimality for $\frac{\alpha}{2-\alpha}$ in the right-hand side of 2) and the left-hand side of 3), we notice from

$$\lim_{t\to\infty}\frac{\alpha A(t,1)+(1-\alpha)G(t,1)}{J_p(t,1)}=\frac{\alpha(p+1)}{2p}\begin{cases}<1, & p>\dfrac{\alpha}{2-\alpha},\\ >1, & p<\dfrac{\alpha}{2-\alpha},\end{cases}$$

that there exists $T\in(1,\infty)$ such that

$$\alpha A(t,1)+(1-\alpha)G(t,1)<J_P(t,1)$$

for $p>\frac{\alpha}{2-\alpha}$ and $t\in(T,+\infty)$, and

$$\alpha A(t,1)+(1-\alpha)G(t,1)>J_P(t,1)$$

for $t\in(T,+\infty)$. This ends the proof of Theorem 2.1.

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REFERENCES

[1] H. Y. Gao and W. J. Niu, "Sharp Inequalities Related to One-Parameter Mean and Gini Mean," *Journal of Mathematical Inequalities*, Vol. 6, No. 4, 2012, pp. 545-555.

[2] W. S. Cheung and F. Qi, "Logarithmic Convexity of the One-Parameter Mean Values," *Taiwanese Journal of Mathematics*, Vol. 11, No. 1, 2007, pp. 231-237.

[3] M. K. Wang, Y. F. Qiu and Y. M. Chu, "An Optimal Double Inequality among the One-Parameter, Arithmetic and Harmonic Means," *Revue D'Analyse Numerique de Theorie de L'approximation*, Vol. 39, No. 2, 2012, pp. 169-175.

[4] H. N. Hu, G. Y. Tu and Y. M. Chu, "Optimal Bouds for the Seiffert Mean in Terms of One-Parameter Means," *Journal of Applied Mathematics*, Vol. 2012, No. 1, 2012, Article ID: 917120.

[5] B. Y. Long and Y. M. Chu, "Optimal Inequalities for Generalized Logarithmic, Arithmetic and Geometric Mean," *Journal of Inequalities and Applications*, Vol. 2010, No. 1, 2010, Article ID: 806825.

[6] N. G. Zheng, Z. H. Zhang and X. M. Zhang, Schur-Convexity of Two Types of One-Parameter Mean Values in Variables," *Journal of Inequalities and Applications*, Vol. 2007, No. 1, 2007, Article ID: 78175.