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The (2,1)-Total Labeling of $S_{n+1} \vee P_m$ and $S_{n+1} \times P_m$

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Abstract

The (2,1)-total labeling number $\lambda_2^T(G)$ of a graph G is the width of the smallest range of integers that suffices to label the vertices and the edges of G such that no two adjacent vertices have the same label, no two adjacent edges have the same label and the difference between the labels of a vertex and its incident edges is at least 2. In this paper, we studied the upper bound of $\lambda_2^T(G)$ of $S_{n+1} \vee P_m$ and $S_{n+1} \times P_m$.

Keywords: Total Labeling, Join of Graph, Path Graph

1. Introduction

Our terminology and notation will be standard. The reader is referred to [1] for the undefined terms. For a graph G, let $V(G)$, $E(G)$, $\Delta(G)$ and $\delta(G)$ denote, respectively, its vertex set, edge set, maximum degree and minimum degree. We use $N(v)$ to denote the neighborhood of v and let $d(v) = |N_G(v)|$ be the degree of v in G. Let $d(x, y)$ denotes the distance of vertices x, y of G, $\lceil x \rceil$ is the smallest integer greater than x.

Motivated by the Frequency Channel assignment problem. Griggs and Yeh [2] introduced the $L(2,1)$-labeling of graphs. This notion was subsequently extended to a general form, named as $L(p,q)$-labeling of graphs. Let p and q be two nonnegative integers. An $L(p,q)$-labeling of graph G is a function f from its vertex set $V(G)$ to the set $\{0,1,2,\cdots k\}$ for some positive integer k such that $|f(x) - f(y)| \geq p$ if x and y are adjacent, and $|f(x) - f(y)| \geq q$ if x and y are at distance 2. The $L(p,q)$-labeling number $\lambda_{p,q}(G)$ of G is the smallest k such that G has an $L(p,q)$-labeling f with max $\{f(v) \mid v \in V(G)\} = k$.

Whittlesey *et al.* [3] investigated the $L(2,1)$-labeling of incidence graphs. The incidence graph of a graph G is the graph obtained from G by replacing each edge by a path of length 2. The $L(2,1)$-labeling of the incidence graph of G is equivalent to an assignment of integers to each element of $V(G) \cup E(G)$ such that adjacent vertices have different labels, adjacent edges have different labels, and incident vertex and edge have the difference of labels by at least 2. This labeling is called $(2,1)$-total labeling of graphs, which was introduced by Havet and Yu [4], and generalized to $(d,1)$-total labeling form. Let $d \geq 1$ be an integer. A $k-(d,1)$-total labeling of graph G is an integer-valued function f defined on the set $V(G) \cup E(G)$ such that

$$|f(x) - f(y)| \geq \begin{cases} 1, & \text{if vertices } x \text{ and } y \text{ are adjacent;} \\ 1, & \text{if edges } x \text{ and } y \text{ are adjacent;} \\ d, & \text{if vertex } x \text{ incident to edge } y. \end{cases}$$

The $(d,1)$-total labeling number, denoted $\lambda_d^T(G)$, is the least integer k such that G has a $k-(d,1)$-total labeling.

When $d = 1$, the $(1,1)$-total labeling is the well-known total coloring of graphs, which has been intensively studied [5-7].

It was conjectured in [4] that $\lambda_d^T(G) \leq \Delta + 2d - 1$ for each graph G, which extends the well-known Total Coloring Conjecture in which $d = 1$. It was also shown in [4] $\lambda_d^T(G) \leq 2\Delta + d - 1$ for any graph G. The $(d,1)$-total labeling for some kinds of special graphs have been studied, e.g., complete graphs [4], outerplanar graphs for $d = 2$ [8], graphs with a given maximum average degree [9], etc.

In this paper, we studied the $(2,1)$-total labeling of joining graph with star and path $S_{n+1} \vee P_m$, and the cartesian product of star and path $S_{n+1} \times P_m$. The following two lemmas appeared in [4], which are very useful.

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Lemma 1.1. Let G be a graph with maximum degree Δ, then $\lambda_d^T(G) \geq \Delta + d - 1$.

Lemma 1.2. If $\lambda_d^T(G) = \Delta + d - 1$, then the vertices with maximum degree of G must be labeled 0 or $\Delta + d - 1$.

2. The (2,1)-Total Labeling of $S_{n+1} \vee P_m$

Let G_1 and G_2 be two graphs, by starting with a disjoint union of G_1 and G_2, adding edges by joining each vertex of G_1 to each vertex of G_2, we can obtain the join of graph G_1 and G_2, denoted $G_1 \vee G_2$.

Let S_{n+1} be a star with n+1 vertices $v_0, v_1, \cdots v_n$, in which $d(v_0) = n$, we call v_0 the center of S_{n+1}. Let P_m be a path with m vertices $u_1, u_2, \cdots u_m$. Then $G = S_{n+1} \vee P_m$ has following propositions:

1) $\Delta(G) = d(v_0) = m + n$;

2) $d(u_2) = d(u_3) = \cdots = d(u_{m-1}) = n + 3$
$d(u_1) = d(u_m) = n + 2$;

3) $d(v_1) = d(v_2) = \cdots d(v_n) = m + 1$.

For $n = 1, 2$, S_{n+1} is a path, S. M. zhang [10] had studied the $(2,1)$-total labeling of $P_m \vee P_n$. So in the sequel, we only consider the case $n \geq 3$.

Theorem 2.1. Let $G = S_{n+1} \vee P_m$, if $m \geq n + 2$, then $\lambda_2^T(G) = \Delta + 1$.

Proof. By lemma 1.1, it's need to prove $\lambda_2^T(G) \leq \Delta + 1 = m + n + 1$.

Now, we give a $(m + n + 1) - (2,1)$-total labeling of G as follows:

For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, m$, let

$$f(v_i u_j) = (i + j - 1) \bmod (m + 1),$$

$$f(v_0 v_i) = i - 1, \quad f(v_0) = m + n - 1, \quad f(v_i) = m + 2$$

$$f(v_0 u_j) = n + j (j = 1, 2, \cdots, m - 1), \ f(v_0 u_m) = n,$$

$$f(u_1) = n + 3, \ f(u_m) = m + n,$$

$$f(u_j) = j - 2 \ (j = 2, 3, \cdots, m - 1),$$

$$f(u_j u_{j+1}) = \begin{cases} m + n, & 1 \leq j \leq m - 2 \ and \ \ j \ is \ odd \\ m + n + 1, & 1 \leq j \leq m - 2 \ and \ \ j \ is \ even \end{cases}$$

$$f(u_{m-1} u_m) = m + n - 2.$$

It's easy to see that f is a $(m + n + 1) - (2,1)$-total labeling of $S_{n+1} \vee P_m$, so we have $\lambda_2^T(G) \leq \Delta + 1 = m + n + 1$.

Theorem 2.2. Let $G = S_{n+1} \vee P_m$, if $m = n + 1 \geq 7$ then $\lambda_2^T(G) = \Delta + 1$

Proof. By lemma 1.1, it's need to prove $\lambda_2^T(G) \leq \Delta + 1 = 2n + 2$.

Now, we give a $(2n + 2) - (2,1)$-total labeling of G as follows:

For $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n + 1$, let

$$f(v_i u_j) = (i + j - 1) \bmod (n + 2)$$

$$f(v_0 v_i) = i - 1,$$

$$f(v_i) = \begin{cases} 2n + 2, & i = 0, \\ n + 3, & i = 1, 2, \cdots, n. \end{cases}$$

$$f(v_0 u_j) = n + j (j = 1, 2, \cdots n), \ f(v_0 u_{n+1}) = n,$$

$$f(u_j u_{j+1}) = j - 1 \ (j = 1, 2, \cdots, n),$$

$$f(u_{n-k}) = \begin{cases} 2n + 1, & 1 \leq k \leq n - 1 \ and \ \ j \ is \ odd \\ 2n, & 1 \leq k \leq n - 1 \ and \ \ j \ is \ even \end{cases},$$

Let $f(u_n) = 2n - 2, \ f(u_{n+1}) = 2n + 1,$ Since $n \geq 6$, we can see that

$$f(u_n) - f(v_2 u_n) = 2n - 2 - (n + 1) = n - 3 \geq 2,$$

$$f(u_n) - f(v_i) = 2n - 2 - (n + 3) = n - 5 \geq 1, .$$

It's easy to see that f is a $(\Delta + 1) - (2,1)$-total labeling of G for $m = n + 1 \geq 7$.

Theorem 2.3. Let $G = S_{n+1} \vee P_m$, if $4 \leq m = n + 1 \leq 6$, then $\lambda_2^T(G) \leq \Delta + 2$.

Proof. Since $(\Delta + 2) = (2n + 3)$, we can also give a $(2n + 3) - (2,1)$-total labeling of G as follows:

Let $f(v_0) = 2n + 3$, $f(u_{n+1}) = 2n + 1$, and let

$$f(u_{n-k}) = \begin{cases} 2n + 2, & 0 \leq k \leq n - 1 \ and \ \ j \ is \ odd \\ 2n + 1, & 0 \leq k \leq n - 1 \ and \ \ j \ is \ even \end{cases},$$

Then label other edges and vertices of G as the proof of theorem 2.2.

Obviously, f is a $(2n + 3) - (2,1)$-total labeling of G, so we have $\lambda_2^T(G) \leq \Delta + 2$.

Theorem 2.4. Let $G = S_{n+1} \vee P_m$, if $m = n \geq 6$, then $G = S_{n+1 \lambda_2^T(G) \leq \Delta + 2} \vee P_m$.

Proof. Since $(\Delta + 2) = (2n + 2)$, we can also give a $(2n + 2) - (2,1)$-total labeling of G as follows:

Let $f(v_i u_j) = (i + j - 1) \bmod (n + 2) \ (i, j = 1, 2, \cdots, n)$,

$$f(v_0 v_i) = i - 1, \ \ f(v_0 u_j) = n + j (j = 1, 2, \cdots, n),$$

$$f(v_0) = 2n + 2, \ \ f(v_i) = 2n + 1 (i = 1, 2, \cdots, n),$$

$$f(u_j u_{j+1}) = j - 1 (j = 1, 2, \cdots, n - 1),$$

$$f(u_j) = n + j + 2 \ (j = 1, 2, \cdots, n - 2),$$

$$f(u_{n-1}) = 2n - 3, \ f(u_n) = 2n - 2.$$

For $n \geq 6$, it's easy to see that

$$f(u_{n-1}) - f(v_3 u_{n-1}) = 2n - 3 - (n + 1) = n - 4 \geq 2,$$

Then f is a $(\Delta + 2) - (2,1)$-total labeling of G, this prove the theorem.

Theorem 2.5. Let $G = S_{n+1} \vee P_m$, if $n > m \geq 5$, then $\lambda_2^T(G) = \Delta + 1$.

Proof. For $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$, let $f(v_i u_j) = (i + j - 1) \bmod (n + 1)$, and $f(v_0 u_j) = j - 1$, $f(v_0 v_i) = i - 1$,

$$f(u_j)=\begin{cases} n+3, & j \text{ is odd}, \\ n+4, & j \text{ is even}. \end{cases}$$

$$f(u_j u_{j+1})=\begin{cases} n+1, & j \text{ is odd}, \\ n+6, & j \text{ is even} \end{cases}.$$

Let

$$f(v_0 v_i)=i+m (i=1,2,\cdots,n-1),$$
$$f(v_i)=m+n (i=1,2,\cdots,n-2),$$
$$f(v_0)=m+n+1, \quad f(v_{n-1})=n+2=f(v_n),$$
$$f(v_0 v_n)=m$$

Notice that $n>m\geq 5$, it's easy to prove that f is a $(\Delta+1)-(2,1)$-total labeling of G, this prove the theorem.

3. The (2, 1) -Total Labeling of $S_{n+1}\times P_m$

The cartesian product of graph G and H, denoted $G\times H$, which vertex set and edge set are the follows:
$V(G\times H)=\{(u,v)\mid u\in G, v\in H\}$,
$E(G\times H)=$
$\{(u,v)(u',v')\mid v=v', uu'\in E(G);\ or\ u=u', vv'\in E(H)\}$.

Let $w_{i,j}(i=0,1,\cdots n;\ j=1,2,\cdots,m)$ denote the vertex (v_i,u_j) of the graph $S_{n+1}\times P_m$.

Obviously, for $m=2$, $\Delta(S_{n+1}\times P_m)=n+1=d(w_{0,j})(j=1,2)$, the degree of other vertexes is 2. For $m\geq 3$,

$$\Delta(S_{n+1}\times P_m)=n+2=d(w_{0,j})(j=1,2,\cdots,m),$$

$d(w_{i,1})=d(w_{i,m})=2(i=1,2,\cdots,n)$, the degree of other vertexes are 3.

For $n=1,2$, S_{n+1} is a path, S. M. zhang [11] had studied the $(2,1)$-total labeling of $P_m\times P_n$. So in the sequel, we only consider the case $n\geq 3$.

Theorem 3.1. Let $G=S_{n+1}\times P_m$, then $\lambda_2^T(G)=\Delta+1$.

Proof. By lemma 1.1, it's need to prove $\lambda_2^T(G)\leq\Delta+1$.

Case 1. If $m=2, n\geq 4$, then $\Delta=n+1$.

We give a $(n+2)-(2,1)$-total labeling of G as follows:

By lemma 1.2, we let $f(w_{0,1})=0, f(w_{0,2})=n+2$,

$$f(w_{0,j}w_{i,j})=\begin{cases} i+1, & i=1,2,\cdots,\left\lceil \frac{n}{2}\right\rceil; j=1,2 \\ i+2, & i=\left\lceil \frac{n}{2}\right\rceil+1, \left\lceil \frac{n}{2}\right\rceil+2,\cdots,n-2; j=1,2 \\ i+2, & i=n-1,n;\ j=1, \\ 0 & i=n-1; j=2 \\ 1 & i=n; j=2. \end{cases}$$

$$f(w_{1,1})=5, f(w_{1,2})=0, f(w_{1,1}w_{1,2})=3;$$
$$f(w_{2,1})=1, f(w_{2,2})=0, f(w_{2,1}w_{2,2})=4;$$
$$f(w_{i,1})=1, f(w_{i,2})=2(i=3,4,\cdots n-1),$$
$$f(w_{n,1})=1, f(w_{n,2})=3,$$
$$f(w_{i,1}w_{i,2})=\left\lceil \frac{n}{2}\right\rceil+2(i=0,3,4,\cdots n).$$

For $n\geq 4$, we have

$$f(w_{0,2})-f(w_{0,1}w_{0,2})=n+2-(\left\lceil \frac{n}{2}\right\rceil+2)=n-\left\lceil \frac{n}{2}\right\rceil\geq 2,$$

then f is a $(n+2)-(2,1)$-total labeling of G for $m=2$, $n\geq 4$.

The $5-(2,1)$-total label of $S_4\times P_2$ as follows:

Let $f(w_{0,1})=0, f(w_{0,2})=5, f(w_{1,1})=2, f(w_{1,2})=3;$
$f(w_{2,1})=5, f(w_{2,2})=0, f(w_{3,1})=1, f(w_{3,2})=2;$
$f(w_{0,1}w_{0,2})=3, \ f(w_{1,1}w_{1,2})=5, \ f(w_{2,1}w_{2,2})=3,$
$f(w_{3,1}w_{3,2})=4; \ f(w_{0,1}w_{1,1})=4, \ f(w_{0,2}w_{1,2})=1,$
$f(w_{0,1}w_{3,1})=5, \ f(w_{0,2}w_{3,2})=0;$
$f(w_{0,1}w_{2,1})=f(w_{0,2}w_{2,2})=2.$

Case 2. If $m\geq 3, n\geq 4$, then $\Delta=n+2$.

We can also give a $(n+3)-(2,1)$-total labeling of G as follows:

For $1\leq j\leq m$, let $f(w_{0,j})=\begin{cases} n+3, & j \text{ is even}, \\ 0, & j \text{ is odd}. \end{cases}$

$$f(w_{0,j}w_{i,j})=\begin{cases} i+1, & i=1,2,\cdots\left\lceil \frac{n}{2}\right\rceil; j=1,2,\cdots m, \\ i+3, & i=\left\lceil \frac{n}{2}\right\rceil+1, \left\lceil \frac{n}{2}\right\rceil+2,\cdots n-2; j=1,2,\cdots m \\ i+3, & i=n-1,n; \text{ and } j \text{ is odd}, \\ 0, & i=n-1; \text{and } j \text{ is even}, \\ 1, & i=n; \text{and } j \text{ is even}. \end{cases}$$

$$f(w_{1,j})=\begin{cases} 4, & j \text{ is odd}, \\ 5, & j \text{ is even}. \end{cases}, \quad f(w_{2,j})=\begin{cases} 5, & j \text{ is odd}, \\ 6, & j \text{ is even}. \end{cases}$$

For $1\leq j\leq m-1$,

let $f(w_{i,j}w_{i,j+1})=\begin{cases} 0, & i=1,2; \text{ and } j \text{ is odd}, \\ 1, & i=1,2; \text{ and } j \text{ is even}. \end{cases}$

For $1\leq j\leq m$,

let $f(w_{i,j})=\begin{cases} 1, & i=3,4\cdots n-1; \text{ and } j \text{ is odd}, \\ 2, & i=3,4\cdots n-1; \text{ and } j \text{ is even}. \end{cases}$

$$f(w_{n,j})=\begin{cases} 1, & j \text{ is odd}, \\ 3, & j \text{ is even}. \end{cases}$$

For $1\leq j\leq m-1$,

$$f(w_{i,j}w_{i,j+1}) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2, & i = 0,3,4\cdots n;\ and\ j\ is\ odd, \\ \left\lceil \frac{n}{2} \right\rceil + 3, & i = 0,3,4\cdots n;\ and\ j\ is\ even. \end{cases}$$

If $n \geq 4,\ 1 \leq j \leq m$ and j is even, we can see that

$$f(w_{0,j}) - f(w_{0,j}w_{0,j+1}) = n + 3 - (\left\lceil \frac{n}{2} \right\rceil + 3) \geq 2,$$

$$f(w_{0,j}) - f(w_{2,j}) = n + 3 - 6 \geq 1,$$

then f is a $(n+3)-(2,1)$ -total labeling of $S_{n+1} \times P_m$.

Case 3. If $m \geq n = 3$, then $\Delta = 5$.

We give a $6-(2,1)$-total labeling of $S_4 \times P_m$ as follows:

$$f(w_{0,j}w_{1,j}) = 4(j = 1,2\cdots m),$$

For $1 \leq j \leq m$, let $f(w_{0,j}) = \begin{cases} 0, & j\ is\ odd, \\ 6, & j\ is\ even. \end{cases}$

$$f(w_{0,j}w_{2,j}) = \begin{cases} 5, & j\ is\ odd, \\ 0, & j\ is\ even. \end{cases}$$

$$f(w_{0,j}w_{3,j}) = \begin{cases} 6, & j\ is\ odd, \\ 1, & j\ is\ even. \end{cases}$$

$$f(w_{i,j}) = \begin{cases} 1, & i = 1,2;\ and\ j\ is\ odd, \\ 2, & i = 1,2;\ and\ j\ is\ even. \end{cases}$$

$$f(w_{3,j}) = \begin{cases} 2, & j\ is\ odd, \\ 3, & j\ is\ even. \end{cases}$$

For $1 \leq j \leq m-1$,

let $f(w_{0,j}w_{0,j+1}) = \begin{cases} 2, & j\ is\ odd, \\ 3, & j\ is\ even. \end{cases}$

$$f(w_{1,j}w_{1,j+1}) = \begin{cases} 5, & j\ is\ odd, \\ 6, & j\ is\ even. \end{cases}$$

$$f(w_{2,j}w_{2,j+1}) = \begin{cases} 4, & j\ is\ odd, \\ 6, & j\ is\ even. \end{cases}$$

$$f(w_{3,j}w_{3,j+1}) = \begin{cases} 0, & j\ is\ odd, \\ 5, & j\ is\ even. \end{cases}.$$

It is easy to see that $\lambda_2^T(S_{n+1} \times P_m) = \Delta + 1$. This prove the Theorem.

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