A New Algorithm for Computing the Determinant and the Inverse of a Pentadiagonal Toeplitz Matrix^{*}

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ABSTRACT

An effective numerical algorithm for computing the determinant of a pentadiagonal Toeplitz matrix has been proposed by Xiao-Guang Lv and others [1]. The complexity of the algorithm is (9n + 3). In this paper, a new algorithm with the cost of (4n + 6) is presented to compute the determinant of a pentadiagonal Toeplitz matrix. The inverse of a pentadiagonal Toeplitz matrix is also considered.

Keywords: Pentadiagonal Matrix; Toeplitz Matrix; Determinant; Nonsingular; Inverse

1. Introduction

Pentadiagonal Toeplitz matrix linear systems often occur in several fields such as numerical solution of differential equations, interpolation problems, boundary value problems [1-5], etc. In these areas, the determinants and the inversions of pentadiagonal Toeplitz matrices are considered. In recent years they have become one of the most important and active research field of applied mathematics and computational mathematics increasingly.

In [2], E. Kilic, M. Ei-Mikkawy presented a fast and reliable algorithm with the cost of (11n-17) for evaluating special *n*th-order pentadiagonal Toeplitz determinants. In [1], X.G. Lv, T.Z. Huang, J. Le presented an algorithm with the cost of (9n+3) for calculating the determinant of a pentadiagonal Toeplitz matrix and an algorithm for calculating the inverse of a pentadiagonal Toeplitz matrix.

In this paper, we present new algorithms for computing the determinant and the inverse of an *n*-by-*n* pentadiagonal Toeplitz matrix. The complexity of the algorithms are (4n+6) and (n^2+5n) respectively.

This paper is organized as follows: in Section 2, we present some useful notations and lemmas. In Section 3, we are going to derive new two algorithms. Finally, we give an numerical examples to show the performance of our algorithms in Section 4.

2. Notations and Preliminaries

Definition 2.1 Let $\boldsymbol{B} = (b_{ij})$ be an $n \times n$ matrix. \boldsymbol{B} is called persymmetric if it symmetric about its northeast-southwest diagonal, *i.e.*, $b_{ij} = b_{n-j+1,n-i+1}$ for all *i* and *j*.

Definition 2.2 Form as

$$\boldsymbol{H} = \left(h_{i-j}\right)_{i,j=1}^{n} = \begin{pmatrix} h_0 & h_{-1} & h_{-2} & \cdots & h_{-n+1} \\ h_1 & h_0 & h_{-1} & \cdots & h_{-n+2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ h_{n-2} & \cdots & h_1 & h_0 & h_{-1} \\ h_{n-1} & h_{n-2} & \cdots & h_1 & h_0 \end{pmatrix}$$

is called Toeplitz matrix.

Toeplitz matrices are all persymmetric matrices.

Lemma 2.1 [6] Let H be an $n \times n$ matrix. Then (1) H is persymmetric matrix if and only if

$$\boldsymbol{J}_{\boldsymbol{\mu}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{J}_{\boldsymbol{\mu}}=\boldsymbol{H}$$
:

(2) If **H** is nonsingular Toeplitz matrix, \mathbf{H}^{-1} is also a Toeplitz matrix, where \mathbf{J}_n is the $n \times n$ exchange matrix, *i.e.*, $\mathbf{J}_n = [e_n, e_{n-1}, \dots, e_1]$, e_i is the *i*th column of identity matrix \mathbf{I}_n of order n.

Without loss of generality, we suppose $n \ge 6$ in the paper. By computing simply, we have the following conclusion:

Lemma 2.2 Let \tilde{T} be an $(n-2) \times (n-2)$ Toeplitz matrix



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$$\tilde{\boldsymbol{T}} = \begin{pmatrix} 1 & & & & \\ a & 1 & & & \\ b & a & 1 & & \\ c & b & a & 1 & & \\ d & c & b & a & 1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & d & c & b & a & 1 \end{pmatrix}$$
(2.1)

Then the inverse of \tilde{T} is an $(n-2)\times(n-2)$ Toeplitz matrix, and

$$\tilde{\boldsymbol{T}}^{-1} = \begin{pmatrix} 1 & & & \\ a_1 & 1 & & \\ a_2 & a_1 & 1 & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & a_2 & a_1 & 1 \end{pmatrix}$$

where

$$a_i = -(aa_{i-1} + ba_{i-2} + ca_{i-3} + da_{i-4}), i = 1, 2, \dots, n-3,$$

and $a_0 = 1, a_{-1} = a_{-2} = a_{-3} = 0$.

Lemma 2.3 Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$, where A, B, C and

0 are matrices of size

 $(n-2)\times(n-2),(n-2)\times2,2\times(n-2),2\times2$ respectively. *A* is nonsingular. Then *M* is nonsingular if and only if $D = CA^{-1}B$ is nonsingular, and

$$\boldsymbol{M}^{-1} = \begin{pmatrix} \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1} \boldsymbol{B} \boldsymbol{D}^{-1} \boldsymbol{C} \boldsymbol{A}^{-1} & \boldsymbol{A}^{-1} \boldsymbol{B} \boldsymbol{D}^{-1} \\ \boldsymbol{D}^{-1} \boldsymbol{C} \boldsymbol{A}^{-1} & -\boldsymbol{D}^{-1} \end{pmatrix}.$$

In the current paper, we consider the $n \times n$ pentadiagonal Toeplitz matrix of the form

3. Main Results

Decompose the pentadiagonal Toeplitz matrix T (2.2) as the following:

$$T = MP$$
,

where

$$\boldsymbol{M} = \begin{pmatrix} 1 & & & a & b \\ a & 1 & & & b & c \\ b & a & 1 & & & c & d \\ c & b & a & 1 & & d & 0 \\ d & c & b & a & 1 & & 0 & 0 \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & d & c & b & a & 1 & 0 & 0 \\ & & d & c & b & a & 0 & 0 \\ & & d & c & b & a & 0 & 0 \\ & & & d & c & b & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{P} = [\boldsymbol{e}_n, \boldsymbol{e}_{n-1}, \boldsymbol{e}_1 \cdots, \boldsymbol{e}_{n-2}].$$

Partition \boldsymbol{M} into $\begin{pmatrix} \tilde{\boldsymbol{T}} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{0} \end{pmatrix}$, where $\tilde{\boldsymbol{T}}$ is matrix (2.1),
 $\boldsymbol{0}$ is zero matrix of size 2×2 ,
 $\begin{pmatrix} a & b & c & d & 0 & \cdots & 0 \end{pmatrix}^{\mathrm{T}}$

$$B = \begin{pmatrix} a & b & c & a & 0 & \cdots & 0 \\ b & c & d & 0 & 0 & \cdots & 0 \end{pmatrix} \text{ of size } (n-2) \times 2 \text{ and}$$

$$C = \begin{pmatrix} 0 & \cdots & 0 & d & c & b & a \\ 0 & \cdots & 0 & 0 & d & c & b \end{pmatrix} \text{ of size } 2 \times (n-2).$$
Thus

 $\boldsymbol{D} = \boldsymbol{C}\tilde{\boldsymbol{T}}^{-1}\boldsymbol{B}$

$$= \begin{pmatrix} d & c & b & a \\ 0 & d & c & b \end{pmatrix} \begin{pmatrix} a_{n-6} & a_{n-7} & a_{n-8} & a_{n-9} \\ a_{n-5} & a_{n-6} & a_{n-7} & a_{n-8} \\ a_{n-4} & a_{n-5} & a_{n-6} & a_{n-7} \\ a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} \end{pmatrix} \begin{pmatrix} a & b \\ b & c \\ c & d \\ d & 0 \end{pmatrix}$$
$$= -\begin{pmatrix} d & c & b & a \\ 0 & d & c & b \end{pmatrix} \begin{pmatrix} a_{n-5} & a_{n-4} + aa_{n-5} \\ a_{n-4} & a_{n-3} + aa_{n-4} \\ a_{n-3} & a_{n-2} + aa_{n-3} \\ a_{n-2} & a_{n-1} + aa_{n-2} \end{pmatrix}$$
$$= \begin{pmatrix} a_{n-1} & a_n + aa_{n-1} \\ a_n + aa_{n-1} & a^2 a_{n-1} + 2aa_n + a_{n+1} \end{pmatrix},$$

where

$$a_{i} = -(aa_{i-1} + ba_{i-2} + ca_{i-3} + da_{i-4}),$$

$$i = n - 2, n - 1, n, n + 1.$$

Denote

$$d_1 = a_{n-1}, d_2 = a_n + aa_{n-1}, d_3 = a^2 a_{n-1} + 2aa_n + a_{n+1},$$

$$\Delta = d_1 d_3 - d_2^2.$$

Thus

$$\det(\mathbf{D}) = \Delta = d_1 d_3 - d_2^2 = a_{n-1} a_{n+1} - a_n^2$$

It is noticed that

$$\begin{pmatrix} I_{n-2} & \mathbf{0} \\ -C\tilde{T}^{-1} & I_2 \end{pmatrix} \begin{pmatrix} \tilde{T} & B \\ C & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \tilde{T} & B \\ \mathbf{0} & -C\tilde{T}^{-1}B \end{pmatrix} = \begin{pmatrix} \tilde{T} & B \\ \mathbf{0} & -D \end{pmatrix}.$$

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$$\det\left(\tilde{\boldsymbol{T}}\right) = 1, \det\left(-\boldsymbol{D}\right) = \left(-1\right)^2 \det\left(\boldsymbol{D}\right) = \Delta.$$

We have

$$det(\boldsymbol{T}) = det(\boldsymbol{M}) det(\boldsymbol{P}) = -det(\boldsymbol{M})$$
$$= -det(\tilde{\boldsymbol{T}}) det(-\boldsymbol{D}) = -\Delta = a_n^2 - a_{n-1}a_{n+1}$$

According to the Lemma 2.3 and deduction above, we have the following results:

Theorem 3.1 Let T be the pentadiagonal Toeplitz matrix as (2.2), then (1) T is nonsingular if and only if $\Delta = a_{n-1}a_{n+1} - a_n^2 \neq 0$, and

(2)
$$\det(\mathbf{T}) = a_n^2 - a_{n-1}a_{n+1}$$
.

 $\tilde{\boldsymbol{M}}_2 = \tilde{\boldsymbol{M}}\tilde{\boldsymbol{D}} = (t_{ij})_{(n-2)\times 2}$. We have

When $\Delta \neq 0$, we have that **T** is nonsingular, and

$$\boldsymbol{T}^{-1} = \boldsymbol{P}^{-1}\boldsymbol{M}^{-1} = \tilde{\boldsymbol{P}}\boldsymbol{M}^{-1},$$

where

$$\tilde{\boldsymbol{P}} = \boldsymbol{P}^{-1} = \begin{bmatrix} e_3, e_4, \cdots, e_{n-2}, e_2, e_1 \end{bmatrix}, \\ \boldsymbol{M}^{-1} = \begin{pmatrix} \tilde{\boldsymbol{T}}^{-1} - \tilde{\boldsymbol{T}}^{-1} \boldsymbol{B} \boldsymbol{D}^{-1} \boldsymbol{C} \tilde{\boldsymbol{T}}^{-1} & \tilde{\boldsymbol{T}}^{-1} \boldsymbol{B} \boldsymbol{D}^{-1} \\ \boldsymbol{D}^{-1} \boldsymbol{C} \tilde{\boldsymbol{T}}^{-1} & -\boldsymbol{D}^{-1} \end{pmatrix},$$

and $\boldsymbol{D} = \boldsymbol{C}\tilde{\boldsymbol{T}}^{-1}\boldsymbol{B}$.

Denote $\tilde{\boldsymbol{M}} = \tilde{\boldsymbol{T}}^{-1}\boldsymbol{B} = \left(s_{ij}\right)_{(n-2)\times 2}, \tilde{\boldsymbol{D}} = \Delta \boldsymbol{D}^{-1}$ and

$$\begin{split} \tilde{\boldsymbol{M}} &= \tilde{\boldsymbol{T}}^{-1} \boldsymbol{B} \\ &= \begin{pmatrix} 1 & & & \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ a_3 & a_2 & a_1 & 1 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & a_3 & a_2 & a_1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \\ c & d \\ d & 0 \\ \vdots & \vdots & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} \\ a_1 & a_0 & a_{-1} & a_{-2} \\ a_2 & a_1 & a_0 & a_{-1} \\ a_3 & a_2 & a_1 & a_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-4} & a_{n-5} & a_{n-6} \end{pmatrix} \begin{pmatrix} a & b \\ b & c \\ c & d \\ d & 0 \end{pmatrix} \\ &= -\begin{pmatrix} a_1 & a_2 + aa_1 \\ a_2 & a_3 + aa_2 \\ \vdots & \vdots \\ a_{n-2} & a_{n-1} + aa_{n-2} \end{pmatrix}, \end{split}$$

i.e.,

$$s_{i1} = -a_i, s_{i2} = -(a_{i+1} + aa_i),$$

$$i = 1, 2, \cdots, n-2,$$
(3.2)

$$\tilde{\boldsymbol{D}} = \Delta \boldsymbol{D}^{-1} = \Delta \begin{pmatrix} d_1 & d_2 \\ d_2 & d_3 \end{pmatrix}^{-1} = \begin{pmatrix} d_3 & -d_2 \\ -d_2 & d_1 \end{pmatrix},$$
$$\tilde{\boldsymbol{M}}_2 = \tilde{\boldsymbol{M}} \tilde{\boldsymbol{D}} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \\ \vdots & \vdots \\ s_{n-21} & s_{n-22} \end{pmatrix} \begin{pmatrix} d_3 & -d_2 \\ -d_2 & d_1 \end{pmatrix},$$

i.e.,

$$t_{i1} = d_3 s_{i1} - d_2 s_{i2}, t_{i2} = d_1 s_{i2} - d_2 s_{i1},$$

$$i = 1, 2, \cdots, n-2.$$
(3.4)

From the Lemma 2.1 and $\boldsymbol{C} = \boldsymbol{B}^{\mathrm{T}} \boldsymbol{J}_{n-2}$, we have

$$\boldsymbol{C}\tilde{\boldsymbol{T}}^{-1} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{J}_{n-2}\tilde{\boldsymbol{T}}^{-1} = \boldsymbol{B}^{\mathrm{T}}\left(\tilde{\boldsymbol{T}}^{-1}\right)^{\mathrm{T}}\boldsymbol{J}_{n-2}$$
$$= \left(\tilde{\boldsymbol{T}}^{-1}\boldsymbol{B}\right)^{\mathrm{T}}\boldsymbol{J}_{n-2} = \tilde{\boldsymbol{M}}^{\mathrm{T}}\boldsymbol{J}_{n-2}.$$

Thus

$$\tilde{\boldsymbol{T}}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} = \tilde{\boldsymbol{M}}\boldsymbol{D}^{-1} = \frac{1}{\Delta}\tilde{\boldsymbol{M}}_{2},$$
$$\boldsymbol{D}^{-1}\boldsymbol{C}\tilde{\boldsymbol{T}}^{-1} = \boldsymbol{D}^{-1}\tilde{\boldsymbol{M}}^{\mathrm{T}}\boldsymbol{J}_{n-2} = \left(\tilde{\boldsymbol{M}}\boldsymbol{D}^{-1}\right)^{\mathrm{T}}\boldsymbol{J}_{n-2}$$
$$= \frac{1}{\Delta}\tilde{\boldsymbol{M}}_{2}^{\mathrm{T}}\boldsymbol{J}_{n-2},$$
$$\tilde{\boldsymbol{T}}^{-1}\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}\tilde{\boldsymbol{T}}^{-1} = \frac{1}{\Delta}\tilde{\boldsymbol{M}}\tilde{\boldsymbol{M}}_{2}^{\mathrm{T}}\boldsymbol{J}_{n-2},$$

 $\int DD CI = \frac{1}{\Delta} M M_2 J_{n-2}$ (persymmetric matrix).

Denote

$$\tilde{\boldsymbol{M}}_{1} = \Delta \tilde{\boldsymbol{T}}^{-1} - \tilde{\boldsymbol{M}} \tilde{\boldsymbol{M}}_{2}^{\mathrm{T}} \boldsymbol{J}_{n-2}.$$
(3.5)

Then $\tilde{T}^{-1} - \tilde{T}^{-1}BD^{-1}C\tilde{T}^{-1} = \frac{1}{\Delta}\tilde{M}_1.$

According to the deduction above, we can obtain Theorem 3.2:

Theorem 3.2 Let the pentadiagonal Toeplitz matrix T as (2.2) be nonsingular. Then

$$\boldsymbol{T}^{-1} = \boldsymbol{\tilde{P}}\boldsymbol{M}^{-1},$$

where

$$\tilde{\boldsymbol{P}} = \begin{bmatrix} \boldsymbol{e}_3, \boldsymbol{e}_4, \cdots, \boldsymbol{e}_{n-2}, \boldsymbol{e}_2, \boldsymbol{e}_1 \end{bmatrix}, \\ \boldsymbol{M}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \tilde{\boldsymbol{M}}_1 & \tilde{\boldsymbol{M}}_2 \\ \tilde{\boldsymbol{M}}_2^{\mathrm{T}} \boldsymbol{J}_{n-2} & -\tilde{\boldsymbol{D}} \end{pmatrix},$$

 $\Delta, \tilde{\boldsymbol{M}}_1, \tilde{\boldsymbol{M}}_2, \boldsymbol{J}_{n-2}, \tilde{\boldsymbol{D}}$ as above.

Combining with Theorem 3.1 and Theorem 3.2, we obtain the following algorithm:

Algorithm 3.1 (Computing det (T)) (1) Input $a,b,c,d,n,a_0 = 1, a_{-1} = a_{-2} = a_{-3} = 0$; (2) Compute

$$a_{i} = -(aa_{i-1} + ba_{i-2} + ca_{i-3} + da_{i-4}),$$

$$i = 1, 2, \dots, n+1;$$

(3) Compute det $(\mathbf{T}) = a_n^2 - a_{n-1}a_{n+1}$. Algorithm 3.2 (Computing \mathbf{T}^{-1}) (1) Using (3.1), calculate d_1, d_2, d_3, Δ ; (2) Using (3.2), calculate $\tilde{\mathbf{M}} = (s_{ij})_{(n-2)\times 2}$; (3) Using (3.4), calculate $\tilde{\mathbf{M}}_2 = (t_{ij})_{(n-2)\times 2}$; (4) Using (3.3) and (3.5), calculate $\tilde{\mathbf{M}}_1, \tilde{\mathbf{D}}$; (5) Calculate $\mathbf{M}^{-1} = \frac{1}{\Delta} \begin{pmatrix} \tilde{\mathbf{M}}_1 & \tilde{\mathbf{M}}_2 \\ \tilde{\mathbf{M}}_2^T \mathbf{J}_{n-2} & -\tilde{\mathbf{D}} \end{pmatrix}$; (6) Calculate $\mathbf{T}^{-1} = \tilde{\mathbf{P}}\mathbf{M}^{-1}$.

Let us now have a look at the number of multiplications and divisions executed by Algorithm 3.1 and 3.2. For Algorithm 3.1, in Step 2, it takes about 4(n+1) operations. Step 3 amounts to 2 operations. On the whole, we need about (4n+6) operations to compute det(T). Algorithm 3.1 is better than E. Killic's algorithm [2] with the cost of (11n-17) and X.G. Lv's algorithm [1] with the cost of (9n+3). For Algorithm 3.2, in Step 1, it takes 4 operations. Step 2 amounts to (n-2) operations. Step 3 amounts to 4(n-2) operations. The cost of step 4 is about n^2 , we make use of the persymmetric matrix. Therefore, we need about (n^2+5n) operations computing T^{-1} . Our algorithm is better than X.G. Lv's algorithm [1] with the cost of $(\frac{3}{2}n^2+13n)$.

4. An Example

Consider the pentadiagonal Toeplitz matrix as

$$\boldsymbol{T} = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}.$$

By Algorithm 3.1, we have

$$a_1 = -1, a_2 = -1, a_3 = 2, a_4 = 0,$$

 $a_5 = -2, a_6 = 1, a_7 = 1.$

So

$$\det(\boldsymbol{T}) = a_6^2 - a_5 a_7 = 3.$$

Using Algorithm 3.2, we obtain

$$d_1 = -2, d_2 = -1, d_3 = 1, \Delta = -3,$$

$$\tilde{\boldsymbol{M}} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \\ -2 & -2 \\ 0 & 2 \end{pmatrix}, \tilde{\boldsymbol{M}}_2 = \begin{pmatrix} 3 & -3 \\ 0 & 3 \\ -4 & 2 \\ 2 & -4 \end{pmatrix},$$
$$\tilde{\boldsymbol{M}}_2^{\mathrm{T}} \boldsymbol{J}_4 = \begin{pmatrix} 2 & -4 & 0 & 3 \\ -4 & 2 & 3 & -3 \end{pmatrix}, \tilde{\boldsymbol{M}}_1 = \begin{pmatrix} 3 & 0 & -6 & 3 \\ -3 & 3 & 3 & -6 \\ -1 & -1 & 3 & 0 \\ 2 & -1 & -3 & 3 \end{pmatrix},$$
$$\boldsymbol{M}_1^{-1} = \frac{1}{3} \begin{pmatrix} -3 & 0 & 6 & -3 & -3 & 3 \\ 3 & -3 & -3 & 6 & 0 & -3 \\ 1 & 1 & -3 & 0 & 4 & -2 \\ -2 & 1 & 3 & -3 & -2 & 4 \\ -2 & 4 & 0 & -3 & 1 & 1 \\ 4 & -2 & -3 & 3 & 1 & -2 \end{pmatrix}.$$
So

$$\boldsymbol{T}^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -2 & -3 & 3 & 1 & -2 \\ -2 & 4 & 0 & -3 & 1 & 1 \\ -3 & 0 & 6 & -3 & -3 & 3 \\ 3 & -3 & -3 & 6 & 0 & -3 \\ 1 & 1 & -3 & 0 & 4 & -2 \\ -2 & 1 & 3 & -3 & -2 & 4 \end{pmatrix}$$

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