Advances in Linear Algebra & Matrix Theory, 2012, 2, 13-19
http://dx.doi.org/10.4236/alamt.2012.22002 Published Online June 2012 (http://www.SciRP.org/journal/alamt)

Solving the Class Equation $x^d = \beta$ in an Alternating Group for Each $\beta \in C^\alpha \cap H_n^c$ and $n > 1$

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Received March 6, 2012; revised April 1, 2012; accepted May 1, 2012

ABSTRACT

The main purpose of this paper is to solve the class equation $x^d = \beta$ in an alternating group, (*i.e.* find the solutions set $X = \{x \in A_n \mid x^d \in A(\beta)\}$) and find the number of these solutions $|X|$ where β ranges over the conjugacy class $A(\beta)$ in A_n and *d* is a positive integer. In this paper we solve the class equation $x^d = \beta$ in A_n where $\beta \in H_n^c \cap C^\alpha$, for all $n > 1$. H_n^c is the complement set of H_n where $H_n = \{C^\alpha$ of $S_n \mid n > 1$, with all parts α_k of α are different and odd$\}$. C^α is conjugacy class of S_n and form class C^α depends on the cycle type α of its elements If $\lambda \in C^\alpha$ and $\lambda \in H_n \cap C^\alpha$, then C^α splits into the two classes $C^{\alpha\pm}$ of A_n.

Keywords: Alternating Groups; Permutations; Conjugate Classes; Cycle Type; Frobenius Equation

1. Introduction

The Frobenius equation $x^d = \beta$ in finite groups was introduced by *G*. Frobenius and then was studied by many others such as ([1-4]). Where they dealt with some types of finite groups like finite cyclic groups, finite *p*-groups, Wreath products of finite groups, etc. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2\cdots\gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{c(\beta)}$. Therefore $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{c(\beta)})$ is a partition of n and it is call cycle type of β. Let $C^\alpha \subset S_n$ be the set of all elements with cycle type α, then we can determine the conjugate class of $\beta \in S_n$ by using cycle type of β, since each pair of λ and β in S_n are conjugate if they have the same cycle type (see [5]). Therefore, the number of conjugacy classes of S_n is the number of partitions of *n*. However, this is not necessarily true in an alternating group. Let $\beta = (1\,2\,4)$ and $\lambda = (1\,4\,2)$ are two permutations in S_4 we have they are belong to the same conjugate class $C^\alpha = [1,3]$ in S_4 (*i.e.* $C^\alpha(\beta) = C^\alpha(\lambda)$) since

$$\alpha(\beta) = (\alpha_1(\beta), \alpha_2(\beta)) = (1,3) = (\alpha_1(\lambda), \alpha_2(\lambda)) = \alpha(\lambda)$$

that means they have the same cycle type but in fact λ and β are not conjugate in A_4, also let $\beta = (1\,2\,3)(4\,5\,6)(7\,8\,9)$ and $\lambda = (5\,3\,7)(1\,6\,9)(2\,4\,8)$ in S_9 we have they are belong to the same conjugate class $C^\alpha = [3^3]$ in S_4 since $\alpha(\beta) = (3,3,3) = \alpha(\lambda)$ but here they are conjugate in A_9. So from the first and second examples we consider it is not necessarily if two permutations have the same cycle type are conjugate in A_n therefore in this work we discuss in detail the conjugacy classes in an alternating group and we denote to conjugacy class of β in A_n by $A(\beta)$. Also we introduce some theorems to solve the class equation $x^d = \beta$ in A_n where $\beta \in H_n^c \cap C^\alpha$, for all $n > 1$.

1.1. Definition [6]

A partition α is a sequence of nonnegative integers $(\alpha_1, \alpha_2, \cdots)$ with $\alpha_1 \ge \alpha_2 \ge \cdots$ and $\sum_{i=1}^{\infty} \alpha_i < \infty$. The length $l(\alpha)$ and the size $|\alpha|$ of α are defined as

$$l(\alpha) = \max\{i \in N; \alpha_i \ne 0\}$$

and $|\alpha| = \sum_{i=1}^{\infty} \alpha_i$. We set $\alpha \vdash n = \{\alpha \text{ partition}; |\alpha| = n\}$ for $n \in N$. An element of $\alpha \vdash n$ is called a partition of *n*.

1.2. Remark [6]

We only write the non zero components of a partition. Choose any $\beta \in S_n$ and write it as $\gamma_1\gamma_2\cdots\gamma_{c(\beta)}$. With γ_i disjoint cycles of length α_i and $c(\beta)$ is the number of disjoint cycle factors including the 1-cycle of β. Since disjoint cycles commute, we can assume that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{c(\beta)}$. Therefore $\alpha = \left(\alpha_1, \alpha_2, \cdots, \alpha_{c(\beta)}\right)$ is a partition of n and each α_i is called part of α.

1.3. Definition [6]

We call the partition

$$\alpha = \alpha(\beta) = \left(\alpha_1(\beta), \alpha_2(\beta), \cdots, \alpha_{c(\beta)}(\beta)\right)_i$$

the cycle type of β.

1.4. Definition [6]

Let α be a partition of n. We define $C^\alpha \subset S_n$ to be the set of all elements with cycle type α.

1.5. Definition [6]

Let $\beta \in S_n$ be given. We define $c_m = c_m^{(n)} = c_m^{(n)}(\beta)$ to be the number of cycles of length m of β.

1.6. Remarks

1) If $\beta \in C^\alpha$, then we write $C^\alpha = C^\alpha(\beta)$.

2) The relationship between partitions and c_m is as follows: if $\beta \in C^\alpha$ is given then $c_m^{(n)}(\beta) = \left|\{i : \alpha_i = m\}\right|$, (see [6])

3) The cardinality of each $C^\alpha = C^\alpha(\beta)$ can be found as follows: $\left|C^\alpha\right| = \dfrac{n!}{z_{\alpha(\beta)}}$ with $z_{\alpha(\beta)} = \prod_{r=1}^{n} r^{c_r}(c_r)!$ and $c_r = c_r^{(n)}(\beta) = \left|\{i : \alpha_i = r\}\right|$, (see [7]).

4) $C^\alpha(\beta)$ splits into two A_n-classes of equal order iff $n > 1$, and the non-zero parts of $\alpha(\beta)$ are different and odd, in every other case $C^\alpha(\beta)$ does not split, (see [8]).

1.7. Lemma [9]

Let p prime number and $\left[a^r\right]$ a conjugate class of symmetric group. If p does not divide a, then the solutions of $x^p \in \left[a^r\right]$ are:

1) $\left[a^r\right]$, if $1 \le r < p$

2) $\left[a^r\right], \left[(pa), a^{r-p}\right], \left[(pa)^2, a^{r-2p}\right], \cdots, \left[(pa)^m, a^{r-mp}\right]$

if $mp \le r < (m+1)p$

1.8. Lemma [9]

Let p and q be different prime numbers and $\left[a^r\right]$ a conjugate class of symmetric group. If $p|a$ and q does not divide a, then the solutions of $x^{pq} \in \left[a^r\right]$ are:

1) $\left[(pa)^i, (pqa)^j\right]$, where i and j are solutions of the equation $i + qj = \dfrac{r}{p}$ if $p|r$.

2) No solution if p does not divide r.

1.9. Lemma [9]

Let p and q be different prime numbers and $\left[a^r\right]$ a conjugate class in S_n. If p does not divide a and q does not divide a, then the solutions of $x^{pq} \in \left[a^r\right]$ are $\left[a^i, (pa)^j, (qa)^k, (pqa)^l\right]$, where i, j, k and l are non-negative integers and solutions of the equation $i + pj + qk + pql = r$.

2. Conjugacy Class $A(\beta)$ of A_n [10]

Let $\beta \in C^\alpha$, where β is a permutation in an alternating group. We define the $A(\beta)$ conjugacy class of β in A_n by:

$$A(\beta) = \left\{\gamma \in A_n \middle| \gamma = t\beta t^{-1}; \text{for some } t \in A_n\right\}$$
$$= \begin{cases} C^\alpha, & \text{if } \beta \notin H_n \\ C^{\alpha+} \text{ or } C^{\alpha-}, & \text{if } \beta \in H_n \end{cases}$$

where $H_n = \{C^\alpha$ of $S_n | n > 1$, with all parts α_k of α different and odd$\}$.

2.1. Remarks

1) $\beta \in H_n \Rightarrow \beta \in A_n$.

2) $\beta \in C^\alpha \cap H_n^c \cap A_n \Rightarrow A(\beta) = C^\alpha$, where H_n^c is complement of H_n.

3) $\beta \in C^\alpha \cap H_n \Rightarrow \beta \in A_n$ and C^α split into two classes $C^{\alpha\pm}$ of A_n.

4) If $\beta, \lambda \in C^\alpha \cap H_n$, and $\lambda \in C^{\alpha+}$, then

$$A(\beta) = \begin{cases} C^{\alpha+} & \text{if } \beta \underset{A_n}{\approx} \lambda \\ C^{\alpha-} & O \cdot W \end{cases}$$

5) If $n \in \theta = \{1, 2, 5, 6, 10, 14\}$, then for each $\beta \in A_n$, β is conjugate to β^{-1} in $A_n\left(\beta \underset{A_n}{\approx} \beta^{-1}\right)$.

2.2. Definition

Let $F_n = \{C^\alpha$ of $S_n |$ the number of parts α_k of α with the property $\alpha_k \equiv 3 \pmod 4$ is odd$\}$. Then, for each $\beta \in H_n \cap C^\alpha \cap F_n$, $C^{\alpha\pm}$ of A_n is defined by

$C^{\alpha+} = \left\{\lambda \in A_n \middle| \lambda = \gamma\beta\gamma^{-1}; \text{for some } \gamma \in A_n\right\} = A(\beta)$,

$C^{\alpha-} = \left\{\lambda \in A_n \middle| \lambda = \gamma\beta^{-1}\gamma^{-1}; \text{for some } \gamma \in A_n\right\} = A(\beta^{-1})$.

2.3. Definition

Let $\overline{F_n} = \{C^\alpha$ of $S_n \mid$ the number of parts α_k of α with the property $\alpha_k \equiv 3$ (mod 4) is even}. Then, for each $\beta \in H_n \cap C^\alpha \cap \overline{F_n}$, $C^{\alpha\pm}$ of A_n is defined by

$C^{\alpha+} = \left\{\lambda \in A_n \middle| \lambda = \gamma\beta\gamma^{-1}; \text{for some } \gamma \in A_n\right\} = A(\beta)$,

$C^{\alpha-} = \left\{\lambda \in A_n \middle| \lambda = \gamma\beta^{-1}\gamma^{-1}; \text{for some } \gamma \in A_n\right\} = A(\beta^{\#})$,

where $\beta^{\#}$ does not conjugate to β.

3. Results for Even Permutations in H_n^c

3.1. Theorem

Let $A(\beta)$ be the conjugacy class of β in A_n. If p is a prime number and does not divide a, $\beta \in [a^r] \cap H_n^c$, where $[a^r]$ is a class of S_n, then the solutions of $x^p \in A(\beta)$ are

1) $[a^r]$ if $(1 \le r < p)$ and (a is odd or (a and r) are even),

2) $[a^r], [(pa), a^{r-p}], [(pa)^2, a^{r-2p}], \cdots, [(pa)^m, a^{r-mp}]$ if [((a and p) are odd) or (p is odd and (a and r) are even)] and $[mp \le r < (m+1)p]$,

3) $[a^r], [(pa)^2, a^{r-2p}], [(pa)^4, a^{r-4p}], \cdots, [(pa)^m, a^{r-mp}]$ if [(a is odd and p is even) or (a, p and r are even)] and [$mp \le r < (m+1)p$ and m is even],

4) $[a^r], [(pa)^2, a^{r-2p}], [(pa)^4, a^{r-4p}], \cdots, [(pa)^{(m-1)}, a^{r-(m-1)p}]$ if [(a is odd and p is even) or (a, p and r are even)] and [$mp \le r < (m+1)p$ and m is odd],

5) $[(pa), a^{r-p}], [(pa)^3, a^{r-3p}], \cdots, [(pa)^m, a^{r-mp}]$ if [(a and p) are even and r is odd] and [$mp \le r < (m+1)p$ and m is odd],

6) $[(pa), a^{r-p}], [(pa)^3, a^{r-3p}], \cdots, [(pa)^{(m-1)}, a^{r-(m-1)p}]$ if [(a and p) are even and r is odd)] and [$mp \le r < (m+1)p$ and m is even], or

7) Does not exist, if [(a is even and (p and r) are odd)].

Proof

Given that $\beta \in [a^r] \cap H_n^c \cap A_n$, $A(\beta) = [a^r]$, then by (1.7), the solutions of $x^p \in A(\beta)$ in S_n are

a) $[a^r]$, if $1 \le r < p$, or $[a^r]$,

b) $[(pa), a^{r-p}], [(pa)^2, a^{r-2p}], \cdots, [(pa)^m, a^{r-mp}]$ if $mp \le r < (m+1)p$.

1) Assume $(1 \le r < p)$ and (a is odd or (a and r) are even), then from a), $[a^r]$ is the solution set of $x^p \in A(\beta)$ in S_n. Let $\lambda \in [a^r]$. If a is odd and $\lambda = \gamma_1\gamma_2\cdots\gamma_r$, where $|\langle\gamma_i\rangle| = a$ (odd) for each $(1 \le i \le r)$, then γ_i is a product of an even number similar to T_i of transpositions for all $(1 \le i \le r)$. For any r (odd or even), λ is a product of $(T_1 + T_2 + \cdots + T_r)$ = (even) number of transpositions $\Rightarrow \lambda \in A_n$. If a and r are even and $\lambda = \gamma_1\gamma_2\cdots\gamma_r$, where $|\langle\gamma_i\rangle| = a$ (even) for each $(1 \le i \le r)$, then γ_i is a product of an odd number similar to T_i of transpositions for all $(1 \le i \le r) \Rightarrow \lambda$ is a product of $(T_1 + T_2 + \cdots + T_r)$ = (even) number of transpositions $\Rightarrow \lambda \in A_n$, then the solution set of $x^p \in A(\beta)$ in A_n is $[a^r]$.

2) Assume [(a and p) are odd) or (p is odd and (a and r) are even)] and $mp \le r < (m+1)p$, then from b),

$[a^r], [(pa), a^{r-p}], [(pa)^2, a^{r-2p}], \cdots, [(pa)^m, a^{r-mp}]$

are solutions of $x^p \in A(\beta)$ in S_n. Let $\lambda \in [a^r] \Rightarrow \lambda \in A_n$, considering that ($a$ is odd or (a and r) are even) and for each $\lambda \in [(pa)^k, a^{r-kp}]$, $(1 \le k \le m)$. If a and p are odd, then $\lambda = \mu\gamma$, where $\gamma = \gamma_1\gamma_2\cdots\gamma_{r-kp}$ and $|\langle\gamma_i\rangle| = a(\text{odd})$, $\forall(1 \le i \le r-kp) \Rightarrow \gamma_i$ is a product of an even number of transpositions for all, $(1 \le i \le r-kp) \Rightarrow \gamma$ is a product of an even number of transpositions, and $\mu = \mu_1\mu_2\cdots\mu_k$, where $|\langle\mu_j\rangle| = ap$ (odd), $\forall(1 \le j \le k) \Rightarrow \mu_j$ is a product of an even number of transpositions for all and $(1 \le j \le k) \Rightarrow \mu$ is a product of an even number of transpositions $\Rightarrow \lambda \in A_n$. If ($p$ is odd and (a and r) are even), then γ_i is a product of an odd number similar to L_i of transpositions, $\forall(1 \le i \le r-kp)$. Moreover, $|\langle\mu_j\rangle| = ap$ (even), and $\forall(1 \le j \le k) \Rightarrow \mu_j$ is a product of an odd number similar to T_j of transpositions for all $(1 \le j \le k)$. If k is odd, then λ is a product of $(L_1 + L_2 + \cdots + L_{r-kp})$ + $(T_1 + T_2 + \cdots + T_k)$ = (odd) + (odd) = (even) number of transpositions $\Rightarrow \lambda \in A_n$. If k is even, then λ is a product of $(L_1 + L_2 + \cdots + L_{r-kp})$ + $(T_1 + T_2 + \cdots + T_k)$ = (even) + (even) = (even) number of transpositions $\Rightarrow \lambda \in A_n$, then the solutions of $x^p \in A(\beta)$ in A_n are

$[a^r], [(pa), a^{r-p}], [(pa)^2, a^{r-2p}], \cdots, [(pa)^m, a^{r-mp}]$.

3) and 4) Assume [(a is odd and p is even) or (a, p, and r are even)] and $\left(mp \le r < (m+1)p\right)$. Then, from b), $\left[a^r\right]$, $\left[(pa), a^{r-p}\right]$, and $\left[(pa)^2, a^{r-2p}\right], \cdots$, $\left[(pa)^m, a^{r-mp}\right]$ are solutions of $x^p \in A(\beta)$ in S_n. Let $\lambda \in \left[a^r\right] \Rightarrow \lambda \in A_n$, [$a$ is odd or (a and r) are even]. For each $\lambda \in \left[(pa)^k, a^{r-kp}\right]$, $(1 \le k \le m)$ if (a is odd and p is even) $\Rightarrow \lambda = \mu\gamma$, where $\gamma = \gamma_1\gamma_2\cdots\gamma_{r-kp}$ and $\left|\langle\gamma_i\rangle\right| = a$ (odd), $\forall(1 \le i \le r-kp) \Rightarrow \gamma_i$ is a product of an even number similar to L_i of transpositions, $(1 \le i \le r-kp)$, and $\mu = \mu_1\mu_2\cdots\mu_k$, where $\left|\langle\mu_j\rangle\right| = ap$ (even), $\forall(1 \le j \le k) \Rightarrow \mu_j$ is a product of an odd number similar to T_j of transpositions for all $(1 \le j \le k)$. If k is odd, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \ldots + T_k\right)$ = (even) + (odd) = (odd) number of transpositions $\Rightarrow \lambda \notin A_n$. If k is even, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \ldots + T_k\right)$ = (even) + (even) = (even) number of transpositions $\Rightarrow \lambda \in A_n$. If ($a$, p, and r are even), then $\lambda = \mu\gamma$, where $\gamma = \gamma_1\gamma_2\cdots\gamma_{r-kp}$ and $\left|\langle\gamma_i\rangle\right| = a$ (even), $\forall(1 \le i \le r-kp) \Rightarrow \gamma_i$ is a product of an odd number similar to L_i of transpositions, $(1 \le i \le r-kp)$ and $\mu = \mu_1\mu_2\cdots\mu_k$, where $\left|\langle\mu_j\rangle\right| = ap$ (even), $\forall(1 \le j \le k) \Rightarrow \mu_j$ is a product of an odd number similar to T_j of transpositions for all $(1 \le j \le k)$. If k is odd, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \ldots + T_k\right)$ = (even) + (odd) = (odd) number of transpositions $\Rightarrow \lambda \notin A_n$. If k is even, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \ldots + T_k\right)$ = (even) + (even) = (even) number of transpositions $\Rightarrow \lambda \in A_n$, then the solutions of $x^p \in A(\beta)$ in A_n are

$$\left[a^r\right], \left[(pa)^2, a^{r-2p}\right], \left[(pa)^4, a^{r-4p}\right], \cdots, \left[(pa)^m, a^{r-mp}\right],$$

(if m is even) and

$$\left[a^r\right], \left[(pa)^2, a^{r-2p}\right], \left[(pa)^4, a^{r-4p}\right], \cdots, \left[(pa)^{(m-1)}, a^{r-(m-1)p}\right]$$

(if m is odd).

5) and 6) Assume [(a and p) are even and r is odd)] and $\left[\left(mp \le r < (m+1)p\right)\right]$. From b), $\Rightarrow$

$$\left[a^r\right], \left[(pa), a^{r-p}\right], \left[(pa)^2, a^{r-2p}\right], \cdots, \left[(pa)^m, a^{r-mp}\right],$$

are solutions of $x^p \in A(\beta)$ in S_n. Let

$$\lambda \in \left[a^r\right] \Rightarrow \lambda = \gamma_1\gamma_2\cdots\gamma_r \Rightarrow \left|\langle\gamma_i\rangle\right| = a \quad \text{(even)},$$

$(1 \le i \le r) \Rightarrow \gamma_i$ is a product of an odd number similar to T_i of transpositions for all, $(1 \le i \le k) \Rightarrow \lambda$ is a product of $\left(T_1 + T_2 + \cdots + T_r\right)$ = (odd) number of transpose-tions, also for each $\lambda \in \left[(pa)^k, a^{r-kp}\right]$, where $(1 \le k \le m) \Rightarrow \lambda = \gamma\beta \Rightarrow \beta = \beta_1\beta_2\cdots\beta_{r-kp}$, where $\left|\langle\beta_i\rangle\right| = a$ (even), $(1 \le i \le r-kp)$ and $\gamma = \gamma_1\gamma_2\cdots\gamma_k$, where $\left|\langle\gamma_j\rangle\right| = pa$ (even) $(1 \le j \le k) \Rightarrow \beta_i$, is product of an odd number similar to L_i of transpositions for each $(1 \le i \le r-kp)$, and γ_j is product of an odd number similar to T_j of transpositions for each $(1 \le j \le k)$. If k is odd, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \cdots + T_k\right)$ = (odd) + (odd) = (even) number of transpositions $\Rightarrow \lambda \in A_n$. If k is even, then λ is a product of $\left(L_1 + L_2 + \cdots + L_{r-kp}\right)$ + $\left(T_1 + T_2 + \cdots + T_k\right)$ = (odd) + (even) = (odd) number of transpositions $\Rightarrow$ $\lambda \notin A_n$. Then, if [(a and p) are even and r is odd)] and $\left(mp \le r < (m+1)p\right)$, then the solutions of $x^p \in A(\beta)$ in A_n are

$$\left[(pa), a^{r-p}\right], \left[(pa)^3, a^{r-3p}\right], \cdots, \left[(pa)^m, a^{r-mp}\right],$$

(if m is odd), or

$$\left[(pa), a^{r-p}\right], \left[(pa)^3, a^{r-3p}\right], \cdots, \left[(pa)^{(m-1)}, a^{r-(m-1)p}\right],$$

(if m is even).

7) Assume (a is even and (p and r) are odd). For each $\lambda \in \left[a^r\right]$ or $\lambda \in \left[(pa)^k, a^{r-kp}\right]$, $(1 \le k \le m) \Rightarrow \lambda \notin A_n$, then there is no solution of $x^p \in A(\beta)$ in A_n.

3.2. Remarks

Let p and q be different prime numbers and $\left[a^r\right]$ a conjugate class of symmetric group. If $p|a$, $p|r$ and q does not divide a we defined collection of sets of conjugate classes of S_n as following:

1) $W = \left\{ \left[(pa)^i, (pqa)^j\right] \middle| i \text{ and } j \text{ are non-negative and solutions of the equation } i + qj = \dfrac{r}{p} \right\}$.

2) $W_1 = \left\{\pi \in W \middle| i + j \text{ is even}\right\}$.

3) $W_2 = \left\{\pi \in W \middle| (i \text{ and } j \text{ are even}) \text{ or } (i \text{ is odd and } j \text{ is even})\right\}$.

4) $W_3 = \left\{\pi \in W \middle| i + j \text{ is odd}\right\}$.

5) $W_4 = \left\{\pi \in W \middle| (i \text{ and } j \text{ are odd}) \text{ or } (i \text{ is even and } j \text{ is odd})\right\}$.

*We note that $W = W_1 \cup W_3$ & $W = W_2 \cup W_4$

** $W_1 \cap W_2 = \left\{\pi \in W \middle| i \text{ and } j \text{ are even}\right\}$,

$W_2 \cap W_4 = \phi$, $W_1 \cap W_3 = \phi$.

$W_1 \cap W_4 = \{\pi \in W | (i \text{ and } j \text{ are odd})\}$,

$W_2 \cap W_3 = \{\pi \in W \mid i \text{ is odd and } j \text{ is even}\}$.

$W_3 \cap W_4 = \{\pi \in W \mid i \text{ is even and } j \text{ is odd}\}$.

3.3. Remarks

1) If a, p and q are odd, then for each $\mu \in \pi$, where $\pi \in W$ we have μ is even.

2) If a is even, then for each $\mu \in \pi$, where $\pi \in W$ we have (μ is even if $\pi \in W_1$) and (μ is odd if $\pi \in W_3$).

3) If p is even, then for each $\mu \in \pi$, where $\pi \in W$ we have (μ is even if $\pi \in W_1$) and (μ is odd if $\pi \in W_3$).

4) If q is even and a, p are odd, then for each $\mu \in \pi$, where $\pi \in W$ we have (μ is even if $\pi \in W_2$) and (μ is odd if $\pi \in W_4$).

3.4. Theorem

Let $A(\beta)$ be a conjugacy class of β in A_n, and $\beta \in [a^r] \cap H_n^c$, where $[a^r]$ is a class of S_n. If p and q are different two prime numbers and $p|a$ and q does not divide a, then the solutions of $x^{pq} \in A(\beta)$ in A_n are:

1) W if $p|r$ and (a, p and q are odd).

2) W_1 if $p|r$ and (a or p is even).

3) W_2 if $p|r$ and (q is even & (a and p) are odd).

4) Not exist if p does not divide r.

5) Not exist if $p|r$, (a or p is even) and $W = W_3$.

6) Not exist if $p|r$, (q is even & (a and p) are odd) and $W = W_4$.

Proof:

Since $\beta \in [a^r] \cap H_n^c \cap A_n$, $A(\beta) = [a^r]$ and by (1.8) we have that the solution of $x^{pq} \in A(\beta)$ in S_n is:

a) W if $p|r$.

b) Not exist if p does not divide r.

1) Assume $p|r$ and (a, p and q are odd). Then from a) we have W is the solution set of $x^{pq} \in A(\beta)$ in S_n. Let $\mu \in \pi$ for each $\pi \in W$, we have μ is even permutation Then the solution set in A_n is W.

2) Assume $p|r$ and (a or p is even). Then from a) we have W is the solution set of $x^{pq} \in A(\beta)$ in S_n. Let $\mu \in \pi$ for each $\pi \in W$, we have (μ is even permutation, if $\pi \in W_1$) and (μ is odd permutation, if $\pi \in W_3$). Then the solution set in A_n is W_1.

3) Assume $p|r$ and (q is even & (a and p) are odd). Then from a) we have W is the solution set of $x^{pq} \in A(\beta)$ in S_n. Let $\mu \in \pi$ for each $\pi \in W$, we have (μ is even permutation, if $\pi \in W_2$) and (μ is odd permutation, if $\pi \in W_4$). Then the solution set in A_n is W_2.

4) Assume p does not divide r. Then from b) we have no solution of $x^{pq} \in A(\beta)$ in $S_n \Rightarrow$ no solution of $x^{pq} \in A(\beta)$ in A_n.

5) and 6) it is clear if $p|r$, (a or p is even) and $W = W_3$, then $W_1 = \phi$ and there exists no solution in A_n, also if $p|r$, (q is even & (a and p) are odd) and $W = W_4$, then $W_2 = \phi$ and there exists no solution in A_n.

3.5. Remarks

Let p and q be two different prime numbers and $[a^r]$ a conjugate class of symmetric group S_n, p does not divide a and q does not divide a we defined a collection of sets of conjugate classes of S_n as following:

$$D = \left\{ \left[a^i, (pa)^j, (qa)^k, (pqa)^l \right] \middle| i, j, k \text{ and } l \text{ are non-negative integers and satisfying the equation } i + jp + kq + lpq = r \right\}.$$

$D_1 = \{\pi \in D | i, j, k \text{ and } l \text{ are all even or all odd}\}$.

$D_2 = \{\pi \in D | i + j \text{ is even, } k + i \text{ is odd and } l + j \text{ is odd}\}$.

$D_3 = \{\pi \in D | i \text{ is odd, } j \text{ is even and } k + l \text{ is odd}\}$.

$D_4 = \{\pi \in D | i \text{ is even, } j \text{ is odd and } k + l \text{ is odd}\}$.

$D_5 = \{\pi \in D | j + l \text{ is even}\}$.

$D_6 = \{\pi \in D | k + l \text{ is even}\}$.

$Q_1 = \{\pi \in D | (i \text{ and } j \text{ are even}) \text{and} (k + l \text{ is odd})\}$.

$Q_2 = \{\pi \in D | (i \text{ is even and } j \text{ is odd}) \text{and} (k + l \text{ is even})\}$.

$Q_3 = \{\pi \in D | (i \text{ is odd and } j \text{ is even}) \text{and} (k + l \text{ is even})\}$.

$Q_4 = \{\pi \in D | (i \text{ and } j \text{ are odd}) \text{and} (k + l \text{ is odd})\}$.

$Q_5 = \{\pi \in D | j + l \text{ is odd}\}$.

$Q_6 = \{\pi \in D | k + l \text{ is odd}\}$.

We can denote D as the following:

- $D = D_1 \cup D_2 \cup D_3 \cup D_4 \cup Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.
- $D = D_5 \cup Q_5$.
- $D = D_6 \cup Q_6$.

3.6. Remarks

1) If a, p and q are odd, then for each $\mu \in \pi$, where

$\pi \in D$, μ is even.

2) If a is even, then for each $\mu \in \pi$, μ is even if $\pi \in D_1 \cup D_2 \cup D_3 \cup D_4$ and μ is odd if $\pi \in Q_1 \cup Q_2 \cup Q_3 \cup Q_4$.

3) If p is even and (a and q) are odd, then for each $\mu \in \pi$, μ is even if $\pi \in D_5$ and μ is odd if $\pi \in Q_5$.

4) If q is even and (a and p) are odd, then for each $\mu \in \pi$, μ is even if $\pi \in D_6$ and μ is odd if $\pi \in Q_6$.

3.7. Theorem

Let $A(\beta)$ be the conjugacy class of β in A_n, and $\beta \in [a^r] \cap H_n^c$, where $[a^r]$ is a class of S_n, p and q are different two prime numbers. If p does not divide a and q does not divide a, then the solution of $x^{pq} \in A(\beta)$ in A_n is

1) D, if a, p and q are odd,
2) Γ, where $\Gamma = D_1 \cup D_2 \cup D_3 \cup D_4$, if a is even,
3) D_5, if a and q are odd, and p is even,
4) D_6, if a and p are odd, and q is even,
5) does not exist, if*a*iseven, and

$$D = Q_1 \cup Q_2 \cup Q_3 \cup Q_4,$$

6) does not exist, if p is even, a and q are odd, and $D = Q_5$, or
7) does not exist, if q is even, a and p are odd, and $D = Q_6$.

Proof

Considering that $\beta \in [a^r] \cap H_n^c \cap A_n$, $A(\beta) = [a^r]$, then by (2.2.11), D is the solution set of $x^{pq} \in A(\beta)$ in S_n.

1) Assume a, p and q are odd. Let $\mu \in \pi$ for each $\pi \in D \Rightarrow \mu$ is even $\Rightarrow \mu \in A_n$. Then the solution set in A_n is D if a, p and q are odd.

2) Assume a is even. Let $\mu \in \pi$ for each $\pi \in D \Rightarrow \mu$ is even $\Rightarrow \mu \in A_n$. Then the solution set in A_n is Γ, if a is even.

3) Assume a and q are odd, and p is even. Let $\mu \in \pi$ for each $\pi \in D \Rightarrow \mu$ is even $\Rightarrow \mu \in A_n$. Then the solution set in A_n is D_5, if a and q are odd, and p is even.

4) Assume a and p are odd, and q is even. Let $\mu \in \pi$ for each $\pi \in D \Rightarrow \mu$ is even $\Rightarrow \mu \in A_n$. Then the solution set in A_n is D_6, if a and p are odd, and q is even.

5) Assume a is even and $D = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \Rightarrow \Gamma = \phi$. Then there is no solution in A_n.

6) and 7) Assume a and q are odd, p is even, and $D = Q_5 \Rightarrow D_5 = \phi$. Then there is no solution in A_n, also if a and p are odd, q is even, and $D = Q_6 \Rightarrow D_6 = \phi$. Then there is no solution in A_n.

3.8. The Number of the Solutions

Assume $\left\{\{\lambda_t\}_{t=1}^{T}, \{\gamma_j\}_{j=1}^{k}, \beta\right\}$ are even permutations where β and $\lambda_t \in H_n^c$ and $\gamma_j \in H_n$ for all $1 \le t \le T$ and $1 \le j \le k$. Then we can find the number of the solutions of class equation $x^d = \beta$ in A_n, where d is a positive integer number as follow:

1) If $\left\{C^\alpha(\lambda_t)\right\}_{t=1}^{T} \cup \left\{C^{\alpha+}(\gamma_j), C^{\alpha-}(\gamma_j)\right\}_{j=1}^{k}$ are the solutions, then the number of solutions set is

$$\sum_{t=1}^{T} \frac{n!}{z_{\alpha(\lambda_t)}} + \sum_{j=1}^{k} \frac{n!}{z_{\alpha(\gamma_j)}},$$

2) If $\left\{C^\alpha(\lambda_t)\right\}_{t=1}^{T}$ are the solutions, then the number of solutions set is

$$\sum_{t=1}^{T} \frac{n!}{z_{\alpha(\lambda_t)}},$$

3) If $\left\{C^{\alpha+}(\gamma_j), C^{\alpha-}(\gamma_j)\right\}_{j=1}^{k}$ are the solutions, then the number of solutions set is

$$\sum_{j=1}^{k} \frac{n!}{z_{\alpha(\gamma_j)}}.$$

3.9. Example

Find the solutions of $x^p \in A(\beta)$ in A_n, and the number of the solutions.

1) If $p = 3$ and $\beta = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$ in A_8.
2) If $p = 2$ and $\beta = (1\ 2\ 3)(4\ 5\ 6)$ in A_6.

Solution:

1) Since $\beta \in [2^4] \cap H_8^c$, $a = 2$, $r = 4$, p does not divide a, $pm \le r < (1+m)p$ where, $m = 1$.

So a and r are even, and p is odd. Then by (3.1) the solutions of $x^3 \in A(\beta)$ in A_8 are $[2^4]$ and $[2,6]$, so the number of solutions is $\frac{(8)!}{2^4(4!)} + \frac{(8)!}{2 \times 6} = 3465$ permutations.

2) Since $\beta \in [3^2] \cap H_6^c$, $a = 3$, $r = 2$, p does not divide a, $pm \le r < (1+m)p$ where, $m = 1$.

Also, since a is odd and p is even. Then by (3.1) the solution set of $x^2 \in A(\beta)$ in A_6 is $[3^2]$. So the number of solutions is $\frac{6!}{3^2 \times 2} = 40$ permutations.

3.10. Remark

If $C^\alpha(\lambda)$ conjugate class of λ in S_n belong to the solution set of class equation $x^d = \beta$ in A_n and $\lambda \in H_n$, then we denote to this set $C^\alpha(\lambda)$ by $C^\alpha(\lambda)^\pm$ or

$\left\{C^{\alpha}(\lambda)^{+}, C^{\alpha}(\lambda)^{-}\right\}$.

3.11. Example

Find the solution of

$$x^{35} \in A((1\ 2\ 5)(6\ 4\ 3)(8\ 10\ 15)(7\ 14\ 11)(13\ 9\ 12))$$

in A_{15} and the number of the solutions.

Solution:

Let $p = 7$ and $q = 5$, since

$$\beta = (1\ 2\ 5)(6\ 4\ 3)(8\ 10\ 15)(7\ 14\ 11)(13\ 9\ 12) \in [3^5] \cap H_{15}^{c}.$$

Then by (3.7) the solutions of

$$x^{35} \in A((1\ 2\ 5)(6\ 4\ 3)(8\ 10\ 15)(7\ 14\ 11)(13\ 9\ 12))$$

in A_{15} are $[3^5]$, $[15]^{+}$ and $[15]^{-}$. So the number of the solutions set is $\frac{(15)!}{3^5(5)!} + \frac{(15)!}{15} = 87223136000$ permutations.

4. Concluding Remarks

By the Cayley's theorem: Every finite group G is isomorphic to a subgroup of the symmetric group S_n, for some $n \geq 1$. Then we can discuss these propositions. Let $x^d = g$ be class equation in finite group G and assume that $f: G \cong A_n$, for some $n > 1$ and $f(g) \in H_n^c \cap C^{\alpha}$. The first question we are concerned with is: What is the possible value of d provided that there is no solution for $x^d = g$ in G? The second question we are concerned with is: what is the possible value of d provided that there is a solution for $x^d = g$ in G? And then we can find the solution and the number of the solution for $x^d = g$ in G by using Cayley's theorem and our theorems in this paper. In another direction, let G be a finite group, and $\pi_i(G) = \{g \in G | i$ the least positive integer number satisfy $g^i = 1\}$. If $|\pi_i(G)| = k_i$, then we write $\pi_i(G) = \{g_{i1}, g_{i2}, \cdots, g_{ik_i}\}$ and $\prod = \{\pi_i(G)\}_{i \geq 1}$. For each $g \in G$ and $g_{ij} \in \pi_i(G)$ we have $(g g_{ij} g^{-1}) = 1$. By the Cayley's theorem we can suppose that $(f: G \cong S_n)$ or $(f: G \cong A_n)$. Also the questions can be summarized as follows:

1) Is $\prod = \{\pi_i(G)\}_{i \geq 1}$ collection set of conjugacy classes of G?

2) Is there some $i \geq 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$ for each C^{α} of A_n, where $(f: G \cong A_n)$?

3) Is there some $i \geq 1$, such that $f^{-1}(C^{\alpha}) = \pi_i(G)$ for each C^{α} of S_n, where $(f: G \cong S_n)$?

4) If $G \cong S_n$ and $p(n)$ is the number of partitions of n, is $|\prod| = p(n)$?

5) If $G \cong A_n$ and A_n has m ambivalent conjugacy classes. It is true that is also necessarily G has m ambivalent conjugacy classes?

Finally we will discuss if there is any relation between F_n, $\overline{F_n}$ and H_n in S_n and what is the possible value of d provided that there is a solution for $x^d = g$ in G where $f(g) \in H_n \cap C^{\alpha}$ and for some n to be:

1) $n \in \theta$.

2) $n \notin \theta$.

REFERENCES

[1] H. Ishihara, H. Ochiai, Y. Takegahara and T. Yoshida, "p-Divisibility of the Number of Solutions of $x^p = 1$ in a Symmetric Group," *Annals of Combinatorics*, Vol. 5, No. 2, 2001, pp. 197-210. doi:10.1007/PL00001300

[2] N. Chigira, "The Solutions of $x^d = 1$ in Finite Groups," *Journal of Algebra*, Vol. 180, No. 3, 1996, pp. 653-661. doi:10.1006/jabr.1996.0086

[3] R. Brauer, "On a Theorem of Frobenius," *American Mathematical Monthly*, Vol. 76, No. 1, 1969, pp. 12-15. doi:10.2307/2316779

[4] Y. G. Berkovich, "On the Number of Elements of Given Order in Finite p-Group," *Israel Journal of Mathematics*, Vol. 73, No. 1, 1991, pp. 107-112. doi:10.1007/BF02773429

[5] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli, "Representation Theory of the Symmetric Groups," Cambridge University Press, New York, 2010.

[6] D. Zeindler, "Permutation Matrices and the Moments of Their Characteristic Polynomial," *Electronic Journal of Probability*, Vol. 15, No. 34, 2010, pp. 1092-1118.

[7] J. J. Rotman, "An Introduction to the Theory of Groups," 4th Edition, Springer-Verlag, New York, 1995.

[8] G. D. James and A. Kerber, "The Representation Theory of the Symmetric Group," Addison-Wesley Publishing, Boston, 1984.

[9] S. A. Taban, "Equations in Symmetric Groups," Ph.D. Thesis, University of Basra, Basra, 2007.

[10] S. Mahmood and A. Rajah, "Solving the Class Equation $x^d = \beta$ in an Alternating Group for each $\beta \in H_n \cap C^{\alpha}$ and $n \notin \theta$," *Journal of the Association of Arab Universities for Basic and Applied Sciences*, Vol. 10, No. 1, 2011, pp. 42-50. doi:10.1016/j.jaubas.2011.06.006

