

An Approach for the Construction of Systems That Self-Generate Chaotic Solitons

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ABSTRACT

This paper proposes a method for constructing partial differential equation (PDE) systems with chaotic solitons by using truncated normal forms of an ordinary differential equation (ODE). The construction is based mainly on the fact that the existence of a soliton in a PDE system is equal to that of a homoclinic orbit in a related ODE system, and that chaos of Šilnikov homoclinic type in the ODE imply that the soliton in the PDE changes its profile chaotically along propagation direction. It is guaranteed that the constructed systems can self-generate chaotic solitons without any external perturbation but with constrained wave velocities in a rigorously mathematical sense.

Keywords: Chaotic Solitons; Partial Differential Equation; Homoclinic Orbit

1. Introduction

Chaotic solitons have been a subject of many theoretical papers over the last decades of years. They have been experimentally observed in several physical systems [1-4]. Despite this large amount of effort, many key issues remain open. The central question addressed in this paper is: How can one construct a partial differential equation (PDE) system that self-generates a chaotic solitary-wave pulse that exists in a rigorous sense? There are few rigorous results that addresses this question. In most cases, rigorous results for the generation of chaotic solitons are obtained through perturbations to a known system, e.g., Schrödinger equation and Ginzburg-Landau equation [5-8], that can generate solitons. Unfortunately, this method is not regular since, in practice, it can be applied only to particular examples. Another rigorous approach is that of C. L. Zheng [9]. In this approach one must find variable separation solutions to a generalized $(2 + 1)$ -dimensional Koteweg de-Vries equation by using Bäcklund transformations [10]. The approach can be in nature categorized to perturbation method. A third approach, by Wu *et al.* in [3] and Ricketts *et al.* in [4], is an experimental scheme. The former described a related system that deliberately provokes inherently nonlinear dynamics to produce chaotic soliton oscillations. The latter showed that the cancellation does more than simply allow the faithful propagation of digital pulses: in fact, it can be used as part of electrical oscillator to produce pulses in the first place. Both of these systems are purely electronic; their relative ease of manufacture gives them

many advantages over the “photonic” devices, involving light waves, that currently dominate soliton research. As defined in [8], a chaotic soliton means that the soliton changes its profile chaotically in time or along propagation direction. From the view of mathematics, a soliton of a PDE system corresponds to a special solution of the PDE, or equally, a soliton corresponds to a homoclinic orbit in a related ordinary differential equation (ODE). Furthermore, if such an orbit is of Shilnikov’s type [11], that is, some Shilnikov’s inequalities are satisfied, then the ODE system definitely has chaos of Shilnikov’s type. It is not difficult to imagine that the corresponding soliton in the original PDE system should be a chaotic soliton since its profile impossibly behaves regularly in time. Such a relation between some solutions of ODE and PDE systems is the basis of our constructing PDE systems that self-generate chaotic solitons. On the other hand, from an ODE system with homoclinic orbits, one may construct many PDE systems such that they can self-generate solitons corresponding to the homoclinic orbits in the ODE system. Therefore, the construction must be treated carefully so that the resulting systems via the construction are as desired.

In this paper, we propose a method for constructing a PDE system self-generating chaotic solitons. The construction is based on the truncated normal forms of an autonomous ODE and on the relationship between ODEs and PDEs when solitary waves are under consideration. The constructed system can self-generate a chaotic solitary-wave pulse without any external perturbation. Moreover, the existence of the solitary-wave pulse in the PDE

system is ensured by that of one homoclinic orbit of an ODE system whereas its chaotic behavior (*i.e.*, the soliton changes its profile chaotically along propagation direction) results from the chaos property of the ODE system which is guaranteed by Ši'lnikov's homoclinic theorem [11].

2. A Candidate Class of ODE Systems and Their Principle Homoclinic Orbits

Since finding a soliton in a PDE system can be changed into proving the existence of a homoclinic orbit in a related ODE system, in this section we will introduce a class of autonomous ODE systems as candidates for the construction of PDE systems that can self-generate chaotic solitons. The introduction of the candidate ODE systems is based on amplitude equations which are derived from normal form theory. Then, we discuss the existence of homoclinic orbits in the ODE systems. We do not prove the existence of the homoclinic orbit, and instead obtain some so-called principle homoclinic orbits by applying perturbation theory. Furthermore, we show that such homoclinic orbits are of Ši'lnikov's type, implying that the ODE systems have chaos of Ši'lnikov's type.

We begin by a general theory about the standard form of nonlinear ODE systems. In the analysis of the stability of static states of autonomous nonlinear ODE systems, linear theory produces normal modes that vary like $t^k e^{st}$, where t is time, k is an integer, and $s = \sigma \pm \omega i$ is a root of the characteristic value equation of the linear theory. Consider an autonomous nonlinear ODE system having a polycritical state in which $\sigma = \omega = 0$ for l roots and $\sigma = 0$ with $\omega \neq 0$ for m pairs of complex conjugate roots. Suppose that, for this state, σ is bounded away from zero for all the other roots of the characteristic equation. By using normal form theory, we can obtain amplitude equations of the form:

$$x' = J_\mu x + g_\mu(x) \tag{1}$$

where $x = (x_1, x_2, \dots, x_d)^T$ with $d = l + 2m$ is the amplitude vector, g_μ is a strictly nonlinear vector-valued function, and $\mu = (\mu_0, \dots, \mu_{n-1})$ is a set of parameters. We restrict ourselves to cases where the number of parameters in the characteristic polynomial for J_μ is equal to n (called criticality). In this case, the characteristic equation (called the critical polynomial) for matrix J_μ may be expressed as

$$\det(J_\mu - sI) = s^n + \sum_{k=0}^{n-1} \mu_k s^k \equiv P_n(\mu; s) = 0 \tag{2}$$

We are interested here in the leading-order terms in g_μ , e.g., for $n = 2$, they are of order x^2 . These leading terms in g_μ are dictated by J_0 . For the critical condi-

tion, $\mu = (\nu, \lambda) = 0$ in the case of $n = 2$, the characteristic equation has zero as a double root. We refer to this case as ξ^2 when J_0 is an irreducible 2×2 matrix. Similarly, we have other ξ^n (see [12]). In addition, when J_μ is in Jordan-Arnold form [12,13], we rewrite Equation (1) as

$$\begin{cases} x'_1 = x_2, x'_2 = x_3, \dots, x'_{n-1} = x_n \\ x'_n + \sum_{l=1}^n \mu_{l-1} x_l = \sum_{m=1}^n \sum_{l=1}^m \alpha_{lm} x_{m+1-l} x_l \equiv g(x_k) \end{cases} \tag{3}$$

where the α_{lm} are constants which can be analytically calculated [14].

Let ε be a small positive number and introduce the scaled variables

$$\tau = \varepsilon t, X_k(\tau) = \varepsilon^{1-n-k} x_k(t), \nu_i = \varepsilon^{i-n} \mu_i \tag{4}$$

where $i = 1, 2, \dots, n-1$. If we arrange matters so that the ν_i are of order unity, the values of μ_i are near to the critical condition $\mu_i = 0$. That is, we position the system in that part of parameter space where, for $i = 1, 2, \dots, n-1$, $\mu_i = O(\varepsilon^{n-i})$. Then Equation (3) becomes

$$\begin{cases} X'_1 = X_2, X'_2 = X_3, \dots, X'_{n-1} = X_n, \\ X'_n + \sum_{l=1}^n \nu_{l-1} X_l = \varepsilon^{-2n} g(\varepsilon^{n+k-1} X_k). \end{cases} \tag{5}$$

If we keep only the leading order terms in g , we may reduce Equation (5) to

$$\frac{d^n X}{d\tau^n} + \sum_{l=0}^{n-1} \nu_l \frac{d^l X}{d\tau^l} = k_1 X^2, \tag{6}$$

where $X = X_1, k_1 = \alpha_{11}$ and we are ignoring terms $O(\varepsilon)$. Once we are within the region of parameter space delimited by $\mu_l = O(\varepsilon^{n-i})$, we may use Equation (6) as the amplitude equation for the configuration ξ^n , for any n , with the error of order ε .

We are interested especially in the case of $n = 3$. In this case, we have the following truncated normal form as the amplitude equation for ξ^3 (see [13,14]):

$$x_i''' = -(\eta - k_4 x) x_i'' - (\nu - k_3 x - k_2 x') x_i' - (\lambda - k_1 x) x \tag{7}$$

where $k_i (1 \leq i \leq 4)$ and ν are constants. Equation (7) has two singular points $x_* = 0$ and $x_* = \lambda/k_1$. We look for solutions of the form $x_* + \xi$. If $|\xi|$ is small, $\xi \propto e^{st}$, and we obtain

$$s^3 + (\eta - k_4 x_*) s^2 + (\nu - k_3 x_* - k_2 s) s + \lambda - 2k_1 x_* = 0 \tag{8}$$

The parameter η is a measure of the dissipation, and assumed to be positive. A change in the sign of λ essentially interchanges the roles of the two singular points, so we need consider only $\lambda < 0$. When $|\nu| \ll |\eta|$ and $|\lambda| \ll |\eta|$, we can obtain some principle homoclinic orbits by applying perturbation theory [12], e.g., for fixed $k_1 = -1, k_2 = k_3 = 0, k_4 = 1$, there is a principle homo-

clinic orbit for ξ^3 when η is small and the Ši'lnikov condition $\nu' < -\lambda' - 2$ and $\lambda' < -1$ is satisfied with $\lambda' = \lambda/\eta^3$, $\nu' = \nu/\eta^2$ (refer **Figure 1**). The region for existence of principle homoclinic orbits constitutes a cusp in shape, which is given by

$$4(1/3 - \nu')^3 = 3(3\lambda' - \nu' + 2/9)^2$$

and the cusp has a vertex at $\nu' = 1/3$, $\lambda' = 1/27$ (see **Figure 1**). Besides these principle homoclinic orbits, there are many other homoclinic orbits in the problem (see [15]). Such a homoclinic orbit will imply the existence of chaos of Ši'lnikov type for an ODE system and ensure the existence of a chaotic soliton for a related PDE (see details in the next section).

3. Constructing a PDE System That Self-Generates a Chaotic Soliton

A particular example is the asymptotic normal form for ξ^3 which is of the form:

$$X_t''' + X_t'' + bX_t' - aX + X^2 = 0 \tag{9}$$

where X has been scaled to give a convenient coefficient to X^2 . Based on such an ODE and inspired by the relation between ODE and PDE when the travelling waves are considered, we construct the following model:

$$\tilde{b} \frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial r^3} + \frac{\partial^2 u}{\partial r^2} - au + u^2 \tag{10}$$

where a and \tilde{b} are positive constants. Assume that

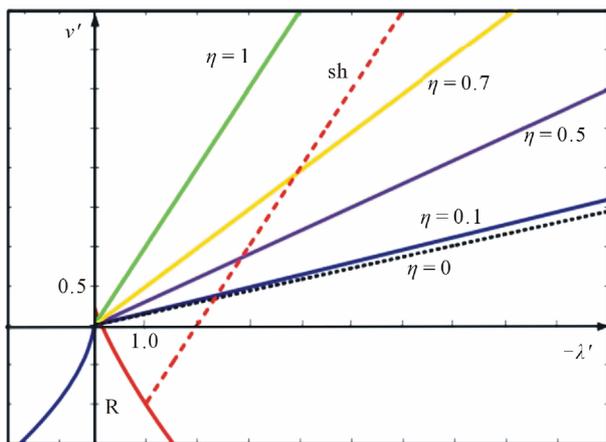


Figure 1. The principle homoclinic conditions for the case $k_1 = -1, k_2 = k_3 = 0, k_4 = 1$ of ξ^3 . The different curves are labelled by the values of η to which they correspond. The dotted curve corresponds to the location of the principle homoclinic orbit of the asymptotic form for ξ^3 (see Equation (1)). Outside the cusp-shaped region R given by $4(1/3 - \nu')^3 = 3(3\lambda' - \nu' + 2/9)^2$, the origin is a saddle-focus. To the right of the red curve marked by Sh, the Ši'lnikov condition $\nu' < -\lambda' - 2$ and $\lambda' < -1$ is satisfied.

Equation (10) has travelling waves of the standard form $u(r,t) \equiv U(\zeta)$, where $\zeta = r - ct$ and c represents travelling wave speed (assume $c > 0$). Then, U satisfies Equation (9) with $b = \tilde{b}c$. Similarly, we can construct the classical Korteweg-de Vries (KdV) equation which has been applied to the optical fibers communication field if we set $\eta = 0$, $\lambda = 0$, $k_1 = k_2 = 0$ and $k_3 = -1, k_4 = 0$, based on the ODE (8). Now, we seek a solitary wave solution, that is, we find the solution of the following equation $U_{\zeta\zeta\zeta}''' + U_{\zeta\zeta}'' + bU_{\zeta}' - aU + U^2 = 0$ subjective to the constrain $U \rightarrow a, U_{\zeta}' \rightarrow 0, U_{\zeta\zeta}'' \rightarrow 0$ as $\zeta \rightarrow \infty$, where the subscript stands for derivatives of function U with respect to the argument ζ . For this, we rewrite Equation (9) into an ODE

$$\begin{cases} \frac{dx}{d\tau} = y \\ \frac{dy}{d\tau} = z \\ \frac{dz}{d\tau} = ax - x^2 - by - z \end{cases} \tag{11}$$

with the condition $x \rightarrow a, y \rightarrow 0$ and $z \rightarrow 0$ as $\tau \rightarrow \infty$, where $b = \tilde{b}c$. Note that finding a soliton solution of Equation (10) satisfying the condition is currently changed into finding a homoclinic orbit of Equation (11) based at the equilibrium point $(a, 0, 0)$.

Fourth-order Rounge-Kutta method is used to solve the system (11) with time step size equal to 0.01, the absolute and relative error equal to 10^{-6} . Numerical simulation shows that the system (11) is chaotic when $a = 3.5$ and $b = 2$, as shown in **Figure 2**. Now, let us fix $a = 3.5$. The numerical simulation also shows that the existence interval of chaos reaches $b \in [1.88, 2.03]$. Thus, we can determine the range of the solitary-wave velocity for fixed \tilde{b} .

We can verify analytically that Equation (10) indeed has a Ši'lnikov homoclinic orbit and further chaos of Ši'lnikov type from a theorem of Ši'lnikov [11]. In fact, we have shown the existence of a homoclinic orbit (e.g., the principle homoclinic orbit) of Ši'lnikov type. **Figure 3** shows the numerical soliton of Equation (10) according to the homoclinic orbit.

In addition, the characteristic polynomial of Jacobian matrix of Equation (10) evaluated at $(a, 0, 0)$ is $\sigma^3 + \sigma^2 + \alpha\sigma + a = 0$ which has three roots: one negative $\sigma_1 = -1.3833$ and a conjugate pair of complex $\sigma_{2,3} = 0.1916 \pm 1.5791i$ with the positive real part for $a = 3.5$ and $\alpha = 2$. A rigorously theoretical proof for the existence of chaos of Ši'lnikov type may be referred to a standard treatment in [16]. A chaotic soliton of Equation (10) with parameter $a = 3.5$ and $\tilde{b} = -5.2$ is shown in **Figure 4** (the numerical simulation is performed for 10^4 iterations with a time step of 0.001).

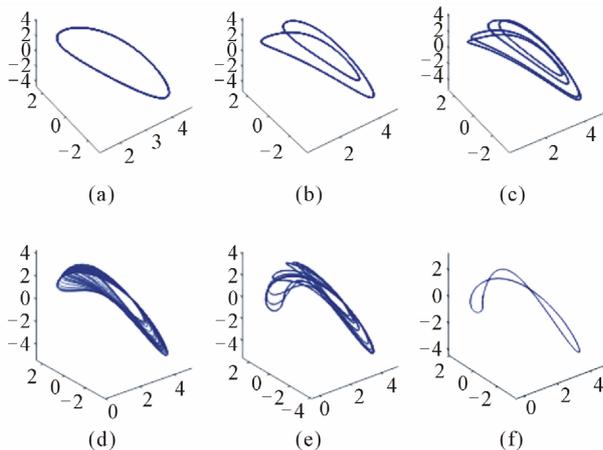


Figure 2. Evolution of the attractor of the asymptotic form for ξ^3 described by Equation (9) or (11) when the parameter b is (a) 2.5; (b) 2.1; (c) 2.05; (d) 2.0; (e) 1.88; (f) 1.419 for fixed $a = 3.5$. Here (a), (b), (c) illustrate the period-doubling cascade leading to the numerical strange attractor shown in (d) and (e). The principle homoclinic orbit is represented in (f): it is unstable. Transients are suppressed.

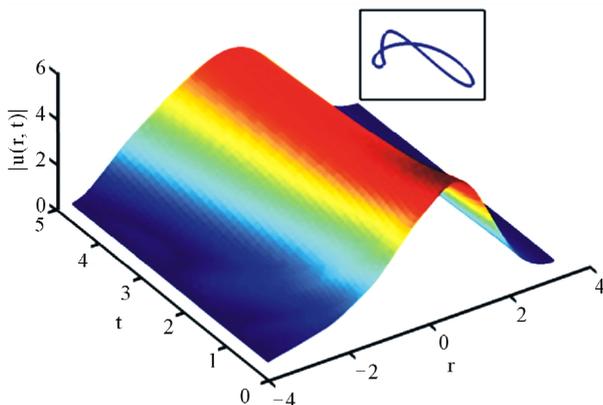


Figure 3. The numerical soliton, where parameter $a = 3.5$, $\tilde{b} = -3.7$.

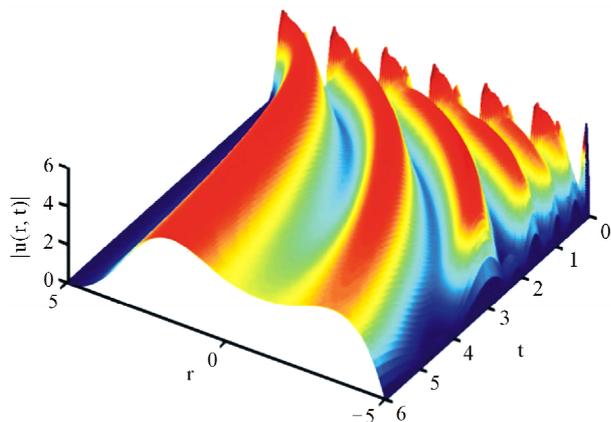


Figure 4. A numerical chaotic soliton, where parameter $a = 3.5$ and $\tilde{b} = -5.2$.

4. Conclusion

Based on normal form theory of ODEs, we have proposed a general method of constructing systems that self-generate chaotic solitons. By adjusting some system parameters in truncated normal forms of ODEs (e.g., Equation (8)), one can construct many systems with chaotic solitons. Note that the constructed system (10) is conservative. It seems the first example that self-generates chaotic solitons.

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