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Solution to Stokes-Maxwell-Euler Differential Equation

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Abstract

Solutions to the differential equation in Smith's Prize Examination taken by Maxwell are discussed. It was a competitive examination using which skill full students were identified and James Clerk Maxwell was one of them. He later formulated the theory of Electromagnetism and predicted the light speed & its value was subsequently confirmed by experiments. Light travel in a direction perpendicular to oscillating electric and magnetic field through a vacuum from sun. In the same exam paper, Maxwell answered the question related to Stokes Theorem of vector calculus which was used in the formalism of Electromagnetic theory.

Keywords

Solution, Differential Equation, Smith's Prize Exam, Stokes, Maxwell, Euler

1. Introduction

Question 6 was a differential equation in the Smith's prize exam. Stokes asked it to integrate. The exam was taken by James Clerk Maxwell at Cambridge in February 1854. Stokes was a personal friend of Maxwell (George Gabriel Stokes [1], EsQ. M.A., was the Lucasian Professor at Cambridge). Maxwell completed the exam and tied for first. There is a solution to this problem in the Mathematical Archives of Leonard Euler [2] [3] [4]. The solution to this differential equation appears with a tangent to a circle whose center is at the origin of the coordinate system $(0,0)$. But our discussion is not the Euler's way of solutions to the problem that is geometrical. But fresh independent solutions that can be given in a mathematics prize exam without knowing what Euler has said before. Solutions were expected from Maxwell by Stokes in the exam. But in the Field of Mathematics, Leonard Euler was another historical figure. On the other hand, Euler's Solutions are in concurrence with us.

2. Smith's Prize Exam

Question 6. Integrate the differential equation

$$\left(a^2-x^2\right)\mathrm{d}y^2+2xy\mathrm{d}x\mathrm{d}y+\left(a^2-y^2\right)\mathrm{d}x^2=0. \tag{1}$$

Solutions:

$$\left(a^2-x^2\right)\left(a^2-y^2\right)\left\{\frac{\mathrm{d}\left(\frac{y}{a}\right)}{\sqrt{1-\left(\frac{y}{a}\right)^2}}\pm\frac{\mathrm{d}\left(\frac{x}{a}\right)}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\right\}^2+2\left\{xy-\pm a^2\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{a}\right)^2+\left(\frac{xy}{a^2}\right)^2}\right\}\mathrm{d}x\mathrm{d}y=0 \tag{2}$$

leads to the solution:

$$\mathrm{Sin}^{-1}\left(\frac{y}{a}\right)\pm\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)=A, x^2+y^2=a^2 \tag{3}$$

due to the reason for right hand side to be zero what are in curly brackets together must be zero.

$$y=a\,\mathrm{Sin}\left(A-\pm\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)\right)=a\,\mathrm{Sin}\,A\,\mathrm{Cos}\left(\pm\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)\right)-\left(\pm\mathrm{Cos}\,A\right)x$$

$$\mathrm{Cos}\,x=1-\frac{x^2}{2}+\frac{x^4}{24}-\frac{x^6}{720}+\cdots$$

$$\mathrm{Sin}\,x=x-\frac{x^3}{6}+\frac{x^5}{120}-\frac{x^7}{5040}+\cdots$$

are the expansions of Cosine and Sine trigonometric functions.

Therefore

$$y=-\left(\pm\mathrm{Cos}\,A\right)x+\left(a\,\mathrm{Sin}\,A\right)=-Bx+C$$

where $C=a\,\mathrm{Sin}\,A, B=\pm\mathrm{Cos}\,A$ to the first order approximation then for A = 0, C= 0, $B=\pm 1$, $y=-(\pm 1)x$,

$$\mathrm{Sin}^{-1}\frac{y}{a}\pm\mathrm{Sin}^{-1}\frac{x}{a}=0, \mathrm{Sin}^{-1}\frac{y}{a}=-\left(\pm\mathrm{Sin}^{-1}\frac{x}{a}\right), \frac{y}{a}=-\pm\mathrm{Sin}\,\mathrm{Sin}^{-1}\frac{x}{a}=-\pm\frac{x}{a}, y=-\pm x$$

two equations of straight lines passing through the origin (0, 0) one inclined at $a^2\,\mathrm{Sin}^2 A$ to the positive side of the x-axis and other inclined 45˚ to the negative side of the x-axis. That is only when A = 0. For non zero A, then $A\neq 0$,

$$x^2+\left(-Bx+C\right)^2=a^2, \quad \left(1+B^2\right)x^2-2BCx+C^2=a^2,$$

$$x^2-\frac{2BC}{1+B^2}x=\frac{a^2-C^2}{1+B^2}, \left(x-\frac{BC}{1+B^2}\right)^2=\frac{a^2-C^2}{1+B^2}+\frac{B^2C^2}{\left(1+B^2\right)^2}=\frac{a^2\left(1+B^2\right)-C^2}{\left(1+B^2\right)^2}$$

$$x=\frac{BC\pm\sqrt{a^2\left(1+B^2\right)-C^2}}{\left(1+B^2\right)^2}, \quad a^2\left(1+B^2\right)>C^2;$$

then two real roots exists cutting the circle by the straight line. The most important observation is if a^2 $(1+B^2)=C^2$ the two real roots coincide and the straight line touch the circle and it becomes a tangent to the circle. But unfortunate fact is this cannot be correct due to the reason $a^2\left(1+\mathrm{Cos}^2 A\right)=\pm a^2\,\mathrm{Sin}^2 A$, $1+\mathrm{Cos}^2 A=\mathrm{Sin}^2 A$ is not a correct result in general but under certain specific conditions it can be true. $\mathrm{Cos}^2 A-\mathrm{Sin}^2 A=-1$, $\mathrm{Cos}\,2A=\mathrm{Cos}\,\pi$, $2A=\pi$, $A=\frac{\pi}{2}$, $\mathrm{Cos}\frac{\pi}{2}=0$, $\mathrm{Sin}\frac{\pi}{2}=1$, $1+0^2=1+0=1=1^2=1$ satisfy the given relation very successfully. The most generalized solution is $A=\pm\frac{\pi}{2}$ so that $\mathrm{Sin}^{-1}\frac{y}{a}\pm\mathrm{Sin}^{-1}\frac{x}{a}=\pm\frac{\pi}{2}$, $y=\pm a$, $x=0$ there are tangents to the circle.

For $a=0$,

$$\left(0^2-x^2\right)\mathrm{d}y^2+2xy\mathrm{d}x\mathrm{d}y+\left(0^2-y^2\right)\mathrm{d}x^2=0$$

$$-\left(x\mathrm{d}y\right)^2+2\left(x\mathrm{d}y\right)\left(y\mathrm{d}x\right)-\left(y\mathrm{d}x\right)^2=0$$

$$\left(x\mathrm{d}y\right)^2-2\left(x\mathrm{d}y\right)\left(y\mathrm{d}x\right)+\left(y\mathrm{d}x\right)^2=0$$

$$\left(x\mathrm{d}y-y\mathrm{d}x\right)^2=0$$

$$x\mathrm{d}y-y\mathrm{d}x=0$$

$$\frac{\mathrm{d}y}{y}-\frac{\mathrm{d}x}{x}=0$$

$$\frac{\mathrm{d}y}{y}=\frac{\mathrm{d}x}{x}$$

$$\int\frac{\mathrm{d}y}{y}=\int\frac{\mathrm{d}x}{x}$$

$$\log_e y=\log_e x+\log_e D$$

by integration

$$\log_e\frac{y}{x}=\log_e D,\ y=Dx$$

is the solution a straight line passes through the origin depending on the value and sign of D a constant. Solutions given by Maxwell are not easily accessible by internet web other than his published volumes of Collected Scientific Papers [5].

$$a\neq 0, x\neq\pm a, \mathrm{d}x\neq 0, \div\left(a^2-x^2\right)\mathrm{d}x^2, \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\frac{2xy}{\left(a^2-x^2\right)}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{\left(a^2-y^2\right)}{\left(a^2-x^2\right)}=0 \quad (4)$$

$$\left[\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{xy}{\left(a^2-x^2\right)}\right]^2=\frac{\left\{x^2y^2-\left(a^2-y^2\right)\left(a^2-x^2\right)\right\}}{\left(a^2-x^2\right)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-xy\pm a\sqrt{x^2+y^2-a^2}}{\left(a^2-x^2\right)}, x^2+y^2\geq a^2 \quad (5)$$

For hold of equality $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}, y\mathrm{d}y=-x\mathrm{d}x$ then by integration $\frac{y^2}{2}=-\frac{x^2}{2}+E, x^2+y^2=F=a^2$ the equation of a circle.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=0,\left(x^2-a^2\right)\left(y^2-a^2\right)=0, y=\pm a, x=0$$

since $x \neq \pm a$

Then by integration, the solution for *y*, a constant, is obtained. There are two tangents to the circle at the origin.

$$a^2\left(\mathrm{d}x^2+\mathrm{d}y^2\right)=\left(x\mathrm{d}y-y\mathrm{d}x\right)^2,\ a\mathrm{d}s=\pm\left(x\mathrm{d}y-y\mathrm{d}x\right)=\pm x^2\mathrm{d}\left(\frac{y}{x}\right)$$

If $x=\pm\alpha$, then by integration $s=\frac{\alpha^2 y}{ax}=\frac{\alpha y}{a}$ the length of the curve under the applied restriction on *x*.

$$\mathrm{d}s^2=\mathrm{d}x^2+\mathrm{d}y^2=\left\{\left[1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]^{1/2}\right\}^2\mathrm{d}x^2,\left(\frac{\mathrm{d}s}{\mathrm{d}x}\right)=\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2},$$

s is the arc length along the curve. $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the limiting tangential gradient to the curve at any point.

$$a\mathrm{d}s=\pm xy\left(\frac{\mathrm{d}y}{y}-\frac{\mathrm{d}x}{x}\right)=\pm xy\left(\mathrm{d}\log_e y-\mathrm{d}\log_e x\right)=\pm xy\left(\mathrm{d}\log_e\left[\frac{y}{x}\right]\right)$$

if

$$xy=k^2, a\mathrm{d}s=\pm k^2\mathrm{d}\log_e\frac{y}{x}$$

then by integration, $s=\left(\frac{k^2}{a}\right)\log_e\left(\frac{y}{x}\right)=\frac{2k^2}{a}\left(\log_e y-\log_e k\right),\frac{y}{x}=\left(\frac{y}{k}\right)^2,x=\frac{k^2}{y}$ is the arc length along the curve under the restriction applied on $x\,y$ product.

α, k are constants assumed. Whether Maxwell answered problem 6 is not stated but he answered the problem 8 of the question paper as well noted which is Stokes Theorem popular in Vector Calculus. What is presented could have been the answers if Maxwell attempted the question 6, these are scattered thoughts. There are seventeen problems. Maxwell answered Stokes Theorem very successfully. So it has become a celebrated proof subsequently.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-xy+a\sqrt{\left(x-y\right)^2+\left(2xy-a^2\right)}}{a^2-x^2}$$

If $2xy=a^2, xy=\frac{a^2}{2},\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\frac{a^2}{2}\pm a\left(x-\frac{a^2}{2x}\right)}{a^2-x^2}$

$$\mathrm{d}y=-\frac{a^2}{2}\frac{\mathrm{d}x}{a^2-x^2}-\pm\frac{a}{2}\frac{-2x\mathrm{d}x}{a^2-x^2}-\pm\frac{a^3}{2}\frac{\mathrm{d}x}{x\left(a^2-x^2\right)}$$

$$=-\frac{a}{4}\left(\frac{\mathrm{d}x}{a+x}-\frac{-\mathrm{d}x}{a-x}\right)-\pm\frac{a}{2}\mathrm{d}\log_e\left(a^2-x^2\right)-\pm\frac{a^3}{2}\frac{\mathrm{d}x}{x\left(a^2-x^2\right)}$$

$$=-\frac{a}{4}\left[\mathrm{d}\log_e\left(a+x\right)-\mathrm{d}\log_e\left(a-x\right)\right]-\pm\frac{a}{2}\mathrm{d}\log_e\left(a^2-x^2\right)-\pm\frac{a}{2}\left[\frac{\mathrm{d}x}{x}-\frac{1}{2}\frac{-2x\mathrm{d}x}{a^2-x^2}\right]$$

$$=-\frac{a}{4}\mathrm{d}\log_e\left(\frac{a+x}{a-x}\right)-\pm\frac{a}{2}\mathrm{d}\log_e\left(a^2-x^2\right)-\pm\frac{a}{2}\mathrm{d}\log_e\frac{x}{\left(a^2-x^2\right)^{1/2}}$$

$$y = -\frac{a}{4}\log_e \frac{a+x}{a-x} - \pm \frac{a}{2}\log_e x\left(a^2 - x^2\right)^{1/2} + G$$

by integration where G is a constant with $y = \frac{a^2}{2x}$, together it provides solutions to x.

$$\text{Sin}^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots$$

the series expansion

$$\text{Sin}^{-1}\frac{y}{a} \pm \text{Sin}^{-1}\frac{x}{a} = \frac{y}{a} \pm \frac{x}{a} = A$$

to the first order approximation.

3. Tangent to a Circle

$$\left(a^2 - x^2\right)\text{d}y^2 + 2xy\text{d}x\text{d}y + \left(a^2 - y^2\right)\text{d}x^2 = 0$$

$$x^2 + y^2 = a^2, y^2 = a^2 - x^2, x^2 = a^2 - y^2$$

$$\left(y\text{d}y\right)^2 + 2\left(y\text{d}y\right)\left(x\text{d}x\right) + \left(x\text{d}x\right)^2 = 0$$

$$\left(y\text{d}y + x\text{d}x\right)^2 = 0$$

$$y\text{d}y + x\text{d}x = 0, \int y\text{d}y + \int x\text{d}x = 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} = K = \frac{a^2}{2}, x^2 + y^2 = a^2$$

by integration. So that equation of a circle exactly fit to the given differential equation, it is the perfect solution and it is circle of radius a whose origin at (0, 0).

$$x^2 + y^2 = a^2$$

$$2x\text{d}x + 2y\text{d}y = 0$$

by differentiation $x\text{d}x + y\text{d}y = 0$, $\frac{\text{d}y}{\text{d}x} = -\frac{x}{y} =$ the gradient of a tangent to the circle & if the point of contact is (x_0, y_0), $\frac{y - y_0}{x - x_0} = -\frac{x_0}{y_0}$ is the equation of a tangent to the circle.

$$yy_0 - y_0^2 = -xx_0 + x_0^2, x_0x + y_0y = x_0^2 + y_0^2 = a^2$$

$$y = -\frac{x_0}{y_0}x + \frac{a^2}{y_0} = -Bx + C, B = \frac{x_0}{y_0}, C = \frac{a^2}{y_0}$$

So that $y = -Bx + C$ is a tangent to the circle $x^2 + y^2 = a^2$ a circle of radius a whose center is at the origin (0, 0) meet the requirements of Euler's formulation of the same problem geometrically. By squaring and reversing $x\text{d}x + y\text{d}y = 0$ it leads to the given differential equation as started. $xy = K^2$ the equation of a rectangular hyperbola. $PV = k$ the Boyle's curve has the same shape the relationship in between pressure and volume of a gas discovered by Robert Boyle in the sixteenth century. $x\text{d}y + y\text{d}x = 0$ by differentiation. Then by squaring

$$x^2\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y + y^2\mathrm{d}x^2 = 0$$

$$a^2 - x^2 = x^2, 2x^2 = a^2, x^2 = \frac{a^2}{2}, x = \pm\frac{a}{\sqrt{2}}$$

$$a^2 - y^2 = y^2, 2y^2 = a^2, y^2 = \frac{a^2}{2}, y = \pm\frac{a}{\sqrt{2}}$$

$x = y = \pm\frac{a}{\sqrt{2}} =$ constants, so it produces the given differential equation

$$\left(a^2 - x^2\right)\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y + \left(a^2 - y^2\right)\mathrm{d}x^2 = 0$$

$$\mathrm{Sin}^{-1}\frac{y}{a} \pm \mathrm{Sin}^{-1}\frac{x}{a} = A$$

$$\frac{\mathrm{d}y}{\sqrt{a^2 - y^2}} \pm \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = 0$$

$$\sqrt{a^2 - x^2}\mathrm{d}y + \sqrt{a^2 - y^2}\mathrm{d}x = 0$$

by squaring

$$\left(a^2 - y^2\right)\mathrm{d}x^2 + 2\sqrt{a^2 - y^2}\sqrt{a^2 - x^2}\mathrm{d}x\mathrm{d}y + \left(a^2 - x^2\right)\mathrm{d}y^2 = 0$$

$$\sqrt{a^2 - x^2} = x, 2x^2 = a^2, x^2 = \frac{a^2}{2}, x = \pm\frac{a}{\sqrt{2}}$$

$$\sqrt{a^2 - y^2} = y, 2y^2 = a^2, y^2 = \frac{a^2}{2}, y = \pm\frac{a}{\sqrt{2}}$$

$$x = y = \pm\frac{a}{\sqrt{2}} = \text{constants}, \frac{x}{a} = \frac{y}{a} = \pm\frac{1}{\sqrt{2}}$$

$$\mathrm{Sin}^{-1}\pm\frac{1}{\sqrt{2}} \pm \mathrm{Sin}^{-1}\pm\frac{1}{\sqrt{2}} = \left(\pm\frac{\pi}{4}\right) \pm \left(\pm\frac{\pi}{4}\right) = \pm\frac{\pi}{2} \text{ or } 0$$

So it produces the differential equation

$$\left(a^2 - x^2\right)\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y + \left(a^2 - y^2\right)\mathrm{d}x^2 = 0$$

$y = \pm x$, a straight line

$$\mathrm{d}y = \pm\mathrm{d}x$$
$$\pm x\mathrm{d}y = \pm y\mathrm{d}x$$
$$x\mathrm{d}y = y\mathrm{d}x$$
$$x\mathrm{d}y - y\mathrm{d}x = 0$$

by squaring

$$x^2\mathrm{d}y^2 - 2xy\mathrm{d}x\mathrm{d}y + y^2\mathrm{d}x^2 = 0$$
$$-x^2\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y - y^2\mathrm{d}x^2 = 0$$

for

$$a = 0, \left(0^2 - x^2\right)\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y + \left(0^2 - y^2\right)\mathrm{d}x^2 = 0$$

so it produces the differential equation

$$\left(a^2 - x^2\right)\mathrm{d}y^2 + 2xy\mathrm{d}x\mathrm{d}y + \left(a^2 - y^2\right)\mathrm{d}x^2 = 0$$

The analysis done would be sufficient.

4. Euler's Geometrical Problem

Euler in 1758 paper (E 236) Explanation of Certain Paradoxes In Integral Calculus, states the same problem as: Given the point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere & solution to it is the given differential equation by Stokes in the Smiths Prize Exam Paper as Question 6 Sat by Maxwell that must be from Euler's Mathematical Literature. Further Stokes asked are there any Singular Solutions?

$x = \pm a, y = \pm a$ are such cases that was questioned by Stokes when the differential equation is attempted for solutions. But no reference to Euler appears in the Exam. Mean while Euler has very long detail solution that Maxwell might have not seen before but Stokes must have known them prior to the Exam problem that was set by him.

5. Conclusion

The results that we have derived are in concurrence with the solutions of Leonard Euler in his archives.

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References

[1] Stokes, G.G. and Maxwell, J.C. (1854) Smith's Prize Exam. http://www.clerkmaxwellfoundation.org/SmithsPrizeExam_Stokes.pdf

[2] Capobianco, G., Enea, M.R. and Ferro, G. (2017) Geometry and Analysis in Euler's Integral Calculus. *Archive for History of Exact Sciences*, **71**, 1-38. http://www.clerkmaxwellfoundation.org/SmithsPrizeExam_Stokes.pdf

[3] Euler, L. (1758) Explanation of Certain Paradoxes in Integral Calculus.

[4] Fabian, A. and Nguyen, H.D. (2013) Paradoxical Euler: Integrating by Differentiating. *The Mathematical Gazette*, **97**, 61-74. http://dx.doi.org/10.1017/S002555720000543X

[5] Maxwell, J.C. (1873) A Dynamical Theory of the Electromagnetic Field. http://archive.org/stream/philtrans00041514/00041514#page/n11/mode/2up