

On Local Existence and Blow-Up of Solutions for Nonlinear Wave Equations of Higher-Order Kirchhoff Type with Strong Dissipation

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Abstract

In this paper, we study on the initial-boundary value problem for nonlinear wave equations of higher-order Kirchhoff type with Strong Dissipation: $u_{tt} + (-\Delta)^m u_t + (a + b \|D^m u\|^{2q})(-\Delta)^m u = |u|^p u$. At first, we prove the existence and uniqueness of the local solution by the Banach contraction mapping principle. Then, by “Concavity” method we establish three blow-up results for certain solutions in the case 1): $E(0) < 0$, in the case 2): $E(0) = 0$ and in the case 3): $E(0) > 0$. At last, we consider that the estimation of the upper bounds of the blow-up time T^{*-} is given for deferent initial energy.

Keywords

Nonlinear Higher-Order Kirchhoff Type Equation, Strong Damping, Local Solutions, Blow-Up, Initial Energy

1. Introduction

In this paper, we are concerned with local existence and blow-up of the solution for nonlinear wave equations of Higher-order Kirchhoff type with strong dissipation:

$$u_{tt} + (-\Delta)^m u_t + (a + b \|D^m u\|^{2q})(-\Delta)^m u = |u|^p u, \quad (x, t) \in \Omega \times [0, +\infty), \quad (1.1)$$

$$u(x, t) = 0, \quad \frac{\partial^i u}{\partial \nu^i} = 0, \quad i = 1, 2, \dots, m-1, \quad x \in \partial\Omega, \quad t \in (0, +\infty), \quad (1.2)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \quad (1.3)$$

where Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and ν

is the unit outward normal on $\partial\Omega$. Moreover, $m > 1$ is an integer constant, and q, p, a and b are some constants such that $q \geq 1, p \geq 0, a \geq 0, b \geq 0$ and $a + b > 0$. We call Equation (1.1) a non-degenerate equation when $a > 0$ and $b > 0$, and a degenerate one when $a = 0$ and $b > 0$. In the case of $a > 0$ and $b = 0$, Equation (1.1) is usual semilinear wave equations.

It is known that Kirchhoff [1] first investigated the following nonlinear vibration of an elastic string for $\delta = f = 0$:

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} = \left\{ p_0 + \frac{Eh}{2L} \int_0^L \left(\frac{\partial u}{\partial x} \right)^2 dx \right\} \frac{\partial^2 u}{\partial x^2} + f; \quad 0 \leq x \leq L, t \geq 0, \tag{1.4}$$

where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t ; ρ : the mass density; h : the cross-section area; L : the length; E : the Young modulus; p_0 : the initial axial tension; δ : the resistance modulus; and f : the external force.

When $a = 1, b = 0, m = 1$, the Equation (1.1) becomes a nonlinear wave equation:

$$u_{tt} - \Delta u - \Delta u_t = |u|^p u, \quad (x, t) \in \Omega \times [0, +\infty), \tag{1.5}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.6}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, +\infty). \tag{1.7}$$

It has been extensively studied and several results concerning existence and blowing-up have been established [2] [3] [4].

When $a > 0, b \geq 0, m = 1$, the Equation (1.1) becomes the following Kirchhoff equation with Lipschitz type continuous coefficient and strong damping:

$$u_{tt} - M(\|\nabla u\|^2) \Delta u - \omega \Delta u_t = |u|^p u, \tag{1.8}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.9}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{1.10}$$

where $\Omega \in R^N, N \geq 1$ is a bounded domain with a smooth boundary $\partial\Omega$. $p > 2$ and $M(s) = m_0 + bs^\gamma$ is a positive local Lipschitz function. Here, $m_0 > 0, b \geq 0, \gamma \geq 1, s \geq 0$. It has been studied and several results concerning existence and blowing-up have been established [5].

When $m = 1$, the Equation (1.1) becomes the following Kirchhoff equation:

$$u_{tt} - \left(a + b \|D^m u\|^{2q} \right) \Delta u - \Delta u_t = |u|^p u, \quad (x, t) \in \Omega \times [0, +\infty), \tag{1.11}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.12}$$

$$u(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, +\infty), \tag{1.13}$$

where Ω is a bounded domain in R^n with the smooth boundary $\partial\Omega$ and ν is the unit outward normal on $\partial\Omega$. Moreover, q, p, a and b are some constants such that $q \geq 1, p \geq 0, a \geq 0, b \geq 0$ and $a + b > 0$. It has been studied and several results concerning existence and blowing-up have been established [6].

When $m = 1$, reference [7] has considered global existence and decay estimates for nonlinear Kirchhoff-type equation:

$$u_{tt} - \varphi(\|\nabla u\|^2)\Delta u - a\Delta u_t = b|u|^{\beta-2}u, \quad (x, t) \in \Omega \times (0, +\infty), \tag{1.14}$$

$$u(x, t) = 0, \quad (x, t) \in \Gamma_1 \times (0, +\infty), \tag{1.15}$$

$$\varphi(\|\nabla u\|^2)\frac{\partial u}{\partial \nu} + a\frac{\partial u_t}{\partial \nu} = g(u_t), \quad (x, t) \in \Gamma_0 \times (0, +\infty), \tag{1.16}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \tag{1.17}$$

where Ω is a bounded domain of R^n ($n \geq 1$) with smooth boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$ and Γ_0, Γ_1 have positive measures, and ν is the unit outward normal on $\partial\Omega$, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative on $\partial\Omega$.

In this paper we shall deal with local existence and blow-up of solutions for nonlinear wave equations of higher-order Kirchhoff type with strong dissipation. The equation may be degenerate or nondegenerate Kirchhoff equation, and derive the blow up properties of solutions of this problem with negative and positive initial energy by the method different from the references [5]-[13].

The content of this paper is organized as follows. In Section 2, we give some lemmas. In Section 3, we prove the existence and uniqueness of the local solution by the Banach contraction mapping principle. In Section 4, we study the blow-up properties of solution for positive and negative initial energy and estimate for blow-up time T^* by lemma of [9].

2. Preliminaries

In this section, we introduce material needed in the proof our main result. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H^m(\Omega)$ with their usual scalar products and norms. Meanwhile we define

$$H_0^m(\Omega) = \left\{ u \in H^m(\Omega) : \frac{\partial^i u}{\partial \nu^i} = 0, i = 0, 1, \dots, m-1 \right\}$$

and introduce the following abbreviations: $\|\cdot\|_{H^m} = \|\cdot\|_{H^m(\Omega)}$, $\|\cdot\|_{H_0^m} = \|\cdot\|_{H_0^m(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$ for any real number $p > 1$.

Lemma 2.1 (Sobolev-Poincaré inequality [8]) Let s be a number with $2 \leq s < +\infty$, $n \leq 2m$ and $2 \leq s \leq \frac{2m}{n-2m}$, $n > 2m$. Then there is a constant K depending on Ω and s such that

$$\|u\|_s \leq K \left\| (-\Delta)^{\frac{m}{2}} u \right\|, \quad \forall u \in H_0^m(\Omega). \tag{2.1}$$

Lemma 2.2 [9] Suppose that $\delta > 0$ and $B(t)$ is a nonnegative $C^2(0, +\infty)$ function such that

$$B''(t) - 4(\delta + 1)B'(t) + 4(\delta + 1)B(t) \geq 0. \tag{2.2}$$

If

$$B'(0) > r_2 B(0) + K_0, \tag{2.3}$$

then we have $\forall t > 0, B'(t) > K_0$. Here, K_0 is a constant and $r_2 = 2(\delta + 1) - 2\sqrt{(\delta + 1)\delta}$ the smallest positive root of the equation $r^2 - 4(\delta + 1)r + 4(\delta + 1) = 0$.

Lemma 2.3 [9] If $J(t)$ is a non-increasing function on $[t_0, +\infty)$, $t_0 \geq 0$ such that

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}}, \quad \forall t_0 \geq 0, \tag{2.4}$$

where $a > 0, b \in \mathbb{R}$. Then there exists a finite time T^* such that $\lim_{t \rightarrow T^{*-}} J(t) = 0$.

Moreover, for the case that $b < 0, J(t_0) < \min\left\{1, \sqrt{\frac{a}{-b}}\right\}$, an upper bound of

$$T^* \text{ is } t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)};$$

If $b = 0$, we have $T^* \leq t_0 + \frac{J(t_0)}{\sqrt{a}}$;

If $b > 0$, we have $T^* \leq \frac{J(t_0)}{\sqrt{a}}$ or $T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left\{1 - [1 + cJ(t_0)]^{\frac{-1}{2\delta}}\right\}$.

3. Local Existence of Solution

Theorem 3.1 Suppose that $0 \leq p \leq \frac{4m}{n-2m}$ ($0 \leq p < +\infty$ if $0 \leq n \leq 2m$) and for any given $(u_0, u_1) \in H^{2m}(\Omega) \cap H_0^m(\Omega) \times L^2(\Omega)$, then there exists $T > 0$ such that the problem (1.1)-(1.3) has a unique local solution satisfying

$$\begin{aligned} u &\in C^0([0, T]; H^{2m}(\Omega) \cap H_0^m(\Omega)), \\ u_t &\in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega)). \end{aligned} \tag{3.1}$$

Proof. We proof the theorem by Banach contraction mapping principle. For $T > 0$ and $R > 0$, we define the following two-parameter space of solutions:

$$X_{T,R} \equiv \left\{ \begin{aligned} &v \in C^0([0, T]; H^{2m}(\Omega) \cap H_0^m(\Omega)), \\ &v_t \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega)), \\ &e_1(v(t)) \leq R^2, t \in [0, T], v(0) = u_0, v_t(0) = u_1 \end{aligned} \right\}, \tag{3.2}$$

where $e_1(v(t)) = \|D^{2m}v\|^2 + \|v_t\|^2$. Then $X_{T,R}$ is a complete metric space with the distance

$$d(v_1, v_2) = \sup_{0 \leq t \leq T} e_1(v_1(t) - v_2(t)). \tag{3.3}$$

We define the non-linear mapping S in the following way. For $v \in X_{T,R}$, $u = Sv$ is the unique solution of the following equation:

$$u_{tt} + (-\Delta)^m u_t + \left(a + b \|D^m v\|^{2q}\right) (-\Delta)^m u = |v|^p v \tag{3.4}$$

$$u(x, t) = 0, \quad \frac{\partial^i u}{\partial v^i} = 0, \quad i = 1, 2, \dots, m-1, \quad x \in \partial\Omega, \quad t \in (0, +\infty), \quad (3.5)$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega. \quad (3.6)$$

We shall show that there exist $T > 0$ and $R > 0$ such that

- 1) S maps $X_{T,R}$ into itself;
- 2) S is a contraction mapping with respect to the metric $d(\cdot, \cdot)$.

First, we shall check (i). Multiplying Equation (3.4) by $2u_t + \frac{2}{3}(-\Delta)^m u$, and

integrating it over Ω , we have

$$\begin{aligned} & \frac{de_2(u(t))}{dt} + \frac{4}{3} \|D^m u_t\|^2 + \frac{2}{3} (a + b \|D^m v\|^{2q}) \|D^{2m} u\|^2 \\ &= \|D^m u\|^2 \frac{d}{dt} (a + b \|D^m v\|^{2q}) + \left(|v|^p v, 2u_t + \frac{2}{3} (-\Delta)^m u \right) \\ &\equiv I_1 + I_2, \end{aligned} \quad (3.7)$$

where $e_2(u(t)) = \|u_t\|^2 + (a + b \|D^m v\|^{2q}) \|D^m u\|^2 + \frac{1}{3} \|D^{2m} u\|^2 + \frac{2}{3} (u_t, D^{2m} u)$.

To proceed the estimation, we observe that for $v \in X_{T,R}$. By **Lemma 2.1**, we have

$$\begin{aligned} I_1 &= \|D^m u\|^2 \frac{d}{dt} (a + b \|D^m v\|^{2q}) \\ &= b \|D^m u\|^2 \frac{d}{dt} \|D^m v\|^{2q} \\ &= 2bq \|D^m u\|^2 \|D^m v\|^{2q-2} (D^{2m} v, v_t) \\ &\leq 2bq K^{2q-2} R^{2q-2} \|D^{2m} v\| \|v_t\| \|D^m u\|^2 \\ &\leq 2bq K^{2q} R^{2q} \|D^{2m} u\|^2 \\ &\leq 2bq K^{2q} R^{2q} e_1(u(t)). \end{aligned} \quad (3.8)$$

Because of $0 \leq p \leq \frac{4m}{n-2m}$ ($0 \leq p < +\infty$ if $0 \leq n \leq 2m$), then

$$\begin{aligned} I_2 &= \left(|v|^p v, 2u_t + \frac{2}{3} (-\Delta)^m u \right) \\ &\leq 2 \left| \int_{\Omega} |v|^p v u_t dx \right| + \frac{2}{3} \left| \int_{\Omega} |v|^p v \cdot D^{2m} u dx \right| \\ &\leq 2 \|u_t\| \|v\|_{2^{p+2}}^{p+1} + \frac{2}{3} \|D^{2m} u\| \|v\|_{2^{p+2}}^{p+1} \\ &\leq 2K^{2p+2} R^{p+1} \|u_t\| + \frac{2}{3} K^{2p+2} R^{p+1} \|D^{2m} u\| \\ &\leq 2K^{2p+2} R^{p+1} (e_1(u(t)))^{\frac{1}{2}} + \frac{2}{3} K^{2p+2} R^{p+1} (e_1(u(t)))^{\frac{1}{2}} \\ &\leq 4K^{2p+2} R^{p+1} (e_1(u(t)))^{\frac{1}{2}}. \end{aligned} \quad (3.9)$$

Since $\frac{2}{3} (u_t, D^{2m} u) \geq -\frac{2}{3} \|u_t\|^2 - \frac{1}{6} \|D^{2m} u\|^2$ by the Young inequality, we see that

$$e_2(u(t)) \geq \frac{1}{3}\|u_t\|^2 + \frac{1}{6}\|D^{2m}u\|^2 \geq \frac{1}{6}e_1(u(t)). \tag{3.10}$$

Combining these inequalities, we get

$$\frac{de_2(u(t))}{dt} + \frac{4}{3}\|D^m u_t\|^2 \leq 12bqK^{2q}R^{2q}e_2(u(t)) + 12K^{2p+2}R^{p+1}(e_2(u(t)))^{\frac{1}{2}}. \tag{3.11}$$

Therefore, by the Gronwall inequality, we obtain

$$e_2(u(t)) + \frac{4}{3}\int_0^t \|D^m u_s\|^2 ds \leq \left[(e_2(u(0)))^{\frac{1}{2}} + 12K^{2p+2}R^{p+1}T \right]^2 e^{12bqK^{2q}R^{2q}T}, \tag{3.12}$$

where $e_2(u(0)) = \|u_1\|^2 + (a+b\|D^m u_0\|^{2q})\|D^m u_0\|^2 + \frac{1}{3}\|D^{2m}u_0\|^2 + \frac{2}{3}(u_1, D^{2m}u_0)$

and

$$e_2(u(0)) \leq 2\|u_1\|^2 + (1+a+b\|D^m u_0\|^{2q})\|D^m u_0\|^2. \tag{3.13}$$

So, for all $t \in [0, T]$, we obtain

$$\begin{aligned} e_1(u(t)) + \frac{4}{3}\int_0^t \|D^m u_s\|^2 ds \\ \leq 6e_2(u(t)) + 8\int_0^t \|D^m u_s\|^2 ds \\ \leq 6 \left[\left(2\|u_1\|^2 + (1+a+b\|D^m u_0\|^{2q})\|D^m u_0\|^2 \right)^{\frac{1}{2}} + 12K^{2p+2}R^{p+1}T \right]^2 e^{12bqK^{2q}R^{2q}T}. \end{aligned} \tag{3.14}$$

Therefore, in order that the map S verifies 1), it will be enough that the parameters T and R satisfy

$$\begin{aligned} 6 \left[\left(2\|u_1\|^2 + (1+a+b\|D^m u_0\|^{2q})\|D^m u_0\|^2 \right)^{\frac{1}{2}} + 12K^{2p+2}R^{p+1}T \right]^2 e^{12bqK^{2q}R^{2q}T} \\ \leq R^2. \end{aligned} \tag{3.15}$$

Moreover, it follows from (3.14) that $u_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega))$ and $u \in L^\infty(0, T; H^{2m}(\Omega) \cap H_0^m(\Omega))$. It implies

$$\begin{aligned} u &\in C^0([0, T]; H^{2m}(\Omega) \cap H_0^m(\Omega)), \\ u_t &\in C^0([0, T]; L^2(\Omega) \cap L^2(0, T; H_0^m(\Omega))). \end{aligned} \tag{3.16}$$

Next, we prove 2). Suppose that (3.15) holds. We take $v_1, v_2 \in X_{T,R}$, let $u_1 = Sv_1$, $u_2 = Sv_2$, and set $w = u_1 - u_2$. Then w satisfies

$$w_t + (-\Delta)^m w_t + (a+b\|D^m v_1\|^{2q})(-\Delta)^m w \tag{3.17}$$

$$\begin{aligned} = -b(\|D^m v_1\|^{2q} - \|D^m v_2\|^{2q})(-\Delta)^m u_2 + (|v_1|^p v_1 - |v_2|^p v_2), \\ (x, t) \in \Omega \times [0, T], \end{aligned} \tag{3.18}$$

$$w(x, t) = \frac{\partial^i w}{\partial v^i} = 0, \quad (x, t) \in \partial\Omega \times [0, T], \tag{3.19}$$

$$w(x, 0) = 0, \quad w_t(x, 0) = 0, \quad x \in \Omega. \tag{3.20}$$

Multiplying (3.17-3.18) by $2w_t$ and integrating it over Ω and using Green's formula, we have

$$\begin{aligned} & \frac{d}{dt} \left[\|w_t\|^2 + (a+b\|D^m v_1\|^{2q}) \|D^m w\|^2 \right] + 2\|D^m w_t\|^2 \\ &= \|D^m w\|^2 \frac{d}{dt} (a+b\|D^m v_1\|^{2q}) - 2b \left(\|D^m v_1\|^{2q} - \|D^m v_2\|^{2q} \right) (D^{2m} u_2, w_t) \\ & \quad + 2 \left(|v_1|^p v_1 - |v_2|^p v_2, w_t \right) \\ & \equiv I_3 + I_4 + I_5. \end{aligned} \quad (3.21)$$

To proceed the estimation, by **Lemma 2.1** observe that

$$I_3 = \|D^m w\|^2 \frac{d}{dt} (a+b\|D^m v_1\|^{2q}) \leq 2bqK^{2q} R^{2q} e_1(w(t)). \quad (3.22)$$

$$\begin{aligned} I_4 &= -2b \left(\|D^m v_1\|^{2q} - \|D^m v_2\|^{2q} \right) (D^{2m} u_2, w_t) \\ & \leq 2bq \left[\theta \|D^m v_1\|^2 + (1-\theta) \|D^m v_2\|^2 \right]^{q-1} \\ & \quad \times \left(\|D^m v_1\| + \|D^m v_2\| \right) \|D^m(v_1 - v_2)\| \|D^{2m} u_2\| \|w_t\| \\ & \leq 4bqK^{2q} R^{2q} \left[e_1(v_1(t) - v_2(t)) \right]^{\frac{1}{2}} \left[e_1(w(t)) \right]^{\frac{1}{2}}, \end{aligned} \quad (3.23)$$

where $0 < \theta < 1$.

$$\begin{aligned} I_5 &= 2 \left(|v_1|^p v_1 - |v_2|^p v_2, w_t \right) \\ & \leq 2 \int_{\Omega} \left(|v_1|^p + |v_2|^p \right) |v_1 - v_2| |w_t| dx \\ & \leq 2 \left(\|v_1\|_{np}^p + \|v_2\|_{np}^p \right) \|v_1 - v_2\|_{\frac{2n}{n-2}} \|w_t\| \\ & \leq 4K^{2p+2} R^p \|D^{2m}(v_1 - v_2)\| \|w_t\| \\ & \leq 4K^{2p+2} R^p \left[e_1(v_1(t) - v_2(t)) \right]^{\frac{1}{2}} \left[e_1(w(t)) \right]^{\frac{1}{2}}. \end{aligned} \quad (3.24)$$

Substituting (3.22)-(3.24) into (3.21), we obtain

$$\begin{aligned} & \frac{d}{dt} \left[\|w_t\|^2 + (a+b\|D^m v_1\|^{2q}) \|D^m w\|^2 \right] + 2\|D^m w_t\|^2 \\ & \leq 2bqK^{2q} R^{2q} e_1(w(t)) \\ & \quad + \left(4bqK^{2q} R^{2q} + 4K^{2p+2} R^p \right) \left[e_1(v_1(t) - v_2(t)) \right]^{\frac{1}{2}} \left[e_1(w(t)) \right]^{\frac{1}{2}}. \end{aligned} \quad (3.25)$$

According to the same method, Multiplying (3.17-3.18) by $2Aw$ and integrating it over Ω , we get

$$\begin{aligned} & \frac{d}{dt} \left[\|D^{2m} w\|^2 + 2(w_t, D^{2m} w) \right] + 2 \left(a+b\|D^m v_1\|^{2q} \right) \|D^{2m} w\|^2 \\ &= 2\|D^m w_t\|^2 - 2b \left(\|D^m v_1\|^{2q} - \|D^m v_2\|^{2q} \right) (D^{2m} u_2, D^{2m} w) \\ & \quad + 2 \left(|v_1|^p v_1 - |v_2|^p v_2, D^{2m} w \right) \\ & \leq 2\|D^m w_t\|^2 + \left(4bqK^{2q} R^{2q} + 4K^{2p+2} R^p \right) \left[e_1(v_1(t) - v_2(t)) \right]^{\frac{1}{2}} \left[e_1(w(t)) \right]. \end{aligned} \quad (3.26)$$

Taking (3.25) $+\frac{1}{3} \times$ (3.26) and by (3.10), it follows that

$$\begin{aligned} & \frac{de_2(w(t))}{dt} \\ & \leq 2bqK^{2q}R^{2q}e_1(w(t)) \\ & \quad + (8bqK^{2q}R^{2q} + 8K^{2p+2}R^p)[e_1(v_1(t) - v_2(t))]^{\frac{1}{2}}[e_1(w(t))]^{\frac{1}{2}} \\ & \leq 2bqK^{2q}R^{2q}e_2(w(t)) \\ & \quad + (72bqK^{2q}R^{2q} + 72K^{2p+2}R^p)[e_2(v_1(t) - v_2(t))]^{\frac{1}{2}}[e_2(w(t))]^{\frac{1}{2}}, \end{aligned} \tag{3.27}$$

where $e_2(w(t)) = \|w_t\|^2 + (a + b\|D^m v_1\|^{2q})\|D^m w\|^2 + \frac{1}{3}\|D^{2m} w\|^2 + \frac{2}{3}(w_t, D^{2m} w)$

and $e_2(w(0)) = 0$.

Applying the Gronwall inequality, we have

$$e_2(w(t)) \leq (72bqK^{2q}R^{2q} + 72K^{2p+2}R^p)^2 T^2 e^{2bqK^{2q}R^{2q}T} \sup_{0 \leq t \leq T} e_1(v_1(t) - v_2(t)). \tag{3.28}$$

So, by (3.10) we have

$$\sup_{0 \leq t \leq T} e_1(u_1(t) - u_2(t)) \leq C_{T,R} \sup_{0 \leq t \leq T} e_1(v_1(t) - v_2(t)), \tag{3.29}$$

where $C_{T,R} \equiv (72\sqrt{6}bqK^{2q}R^{2q} + 72\sqrt{6}K^{2p+2}R^p)^2 T^2 e^{2bqK^{2q}R^{2q}T}$. If $C_{T,R} < 1$, we can see S is a contraction mapping. Finally, we choose suitable R is sufficiently large and T is sufficiently small, such that 1) and 2) hold. By applying Banach fixed point theorem, we obtain the local existence.

4. Blow-Up of Solution

In this section, we shall discuss the blow-up properties for the problem (1.1)-(1.3). For this purpose, we give the following definition and lemmas.

Now, we define the energy function of the solution u of (1.1)-(1.3) by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{a}{2}\|D^m u\|^2 + \frac{b}{2q+2}\|D^m u\|^{2q+2} - \frac{1}{p+2}\|u\|_{p+2}^{p+2}, t \geq 0. \tag{4.1}$$

Then, we have

$$E(t) = E(0) - \int_0^t \|D^m u_t(s)\|^2 ds, \tag{4.2}$$

where $E(0) = \frac{1}{2}\|u_1\|^2 + \frac{a}{2}\|D^m u_0\|^2 + \frac{b}{2q+2}\|D^m u_0\|^{2q+2} - \frac{1}{p+2}\|u_0\|_{p+2}^{p+2}$.

Definition 4.1 A solution $u(t)$ of (1.1)-(1.3) is called a blow-up solution, if there exists a finite time T^* such that

$$\lim_{t \rightarrow T^{*-}} \int_{\Omega} |\nabla^m u|^2 dx = +\infty. \tag{4.3}$$

For the next lemma, we define

$$F(t) := F(u(t)) = \|u(t)\|^2 + \int_0^t \|D^m u(s)\|^2 ds. \tag{4.4}$$

Lemma 4.1 Suppose that $0 \leq p \leq \frac{4m}{n-2m}$ ($0 \leq p < +\infty$ if $0 \leq n \leq 2m$) and

$p \geq 2q$ hold. Then we have the following results, which are

1) $F''(t) - (p+4)\|u_t\|^2 \geq -(4+2p)E(0) + (4+2p)\int_0^t \|D^m u_t(s)\|^2 ds$, for $t \geq 0$;

2) If $E(0) < 0$, we get $F'(t) > \|D^m u_0\|^2$ for $t > t^*$, where

$$t^* = \max \left\{ 0, \frac{F'(0) - \|D^m u_0\|^2}{(4+2p)E(0)} \right\};$$

3) If $E(0) = 0$ and if $F'(0) > \|D^m u_0\|^2$, i.e. $\int_{\Omega} u_0 u_1 dx > 0$ hold, then we have $F'(t) > \|D^m u_0\|^2$ for $t \geq 0$;

4) If $E(0) > 0$ and

$$F'(0) > \left(\frac{4+p-\sqrt{p^2+4p}}{2} \right) \left[\left(F(0) + \frac{(4+2p)E(0) + (4+p)\|D^m u_0\|^2}{4+p} \right) \right] + \|D^m u_0\|^2$$

hold, then we get $F'(t) > \|D^m u_0\|^2$ for $t > 0$.

Proof. Step 1: From (4.4), we obtain

$$F'(t) = 2(u, u_t) + \|D^m u(t)\|^2, \tag{4.5}$$

and

$$\begin{aligned} F''(t) &= 2\|u_t\|^2 + 2(u, u_{tt}) + \frac{d}{dt} \|D^m u\|^2 \\ &= 2\|u_t\|^2 + 2\left(u, -(-\Delta)^m u_t - (a+b\|D^m u\|^{2q})(-\Delta)^m u + |u|^p u\right) + \frac{d}{dt} \|D^m u\|^2 \\ &= 2\|u_t\|^2 - 2\left(a+b\|D^m u\|^{2q}\right)\|D^m u\|^2 + 2\|u\|_{p+2}^{p+2}. \end{aligned} \tag{4.6}$$

From the above equation and the energy identity and $p \geq 2q$, we obtain

$$\begin{aligned} &F''(t) - (4+p)\|u_t\|^2 \\ &= -(p+2)\|u_t\|^2 - 2\left(a+b\|D^m u\|^{2q}\right)\|D^m u\|^2 + 2\|u\|_{p+2}^{p+2} \\ &= -(p+2)\left(2E(0) - 2\int_0^t \|D^m u_t(s)\|^2 ds - a\|D^m u\|^2 - \frac{b}{q+1}\|D^m u\|^{2q+2} + \frac{2}{p+2}\|u\|_{p+2}^{p+2}\right) \\ &\quad - 2\left(a+b\|D^m u\|^{2q}\right)\|D^m u\|^2 + 2\|u\|_{p+2}^{p+2} \\ &= -(2p+4)E(0) + (2p+4)\int_0^t \|D^m u_t(s)\|^2 ds + ap\|D^m u\|^2 + \frac{b(p-2q)}{q+1}\|D^m u\|^{2q+2} \\ &\geq -(2p+4)E(0) + (2p+4)\int_0^t \|D^m u_t(s)\|^2 ds. \end{aligned} \tag{4.7}$$

Therefore, we obtain 1).

Step 2: If $E(0) < 0$, then by (i), we have

$$F''(t) \geq -(2p+4)E(0). \tag{4.8}$$

Integrating (4.8) over $[0, t]$, we have that

$$F'(t) \geq F'(0) - (2p+4)E(0)t, \quad t \geq 0. \tag{4.9}$$

Thus, we get $F'(t) > \|D^m u_0\|^2$ for $t > t^*$, where

$$t^* = \max \left\{ 0, \frac{F'(0) - \|D^m u_0\|^2}{(4 + 2p)E(0)} \right\}.$$

So, 2) has been proved.

Step 3: If $E(0) = 0$, then for $t \geq 0$ we have

$$F''(t) \geq (p + 4)\|u_t\|^2 + (2p + 4) \int_0^t \|D^m u_t(s)\|^2 ds \geq 0. \tag{4.10}$$

Integrating (4.10) over $[0, t]$, we have that

$$F'(t) \geq F'(0). \tag{4.11}$$

And because of $F'(0) > \|D^m u_0\|^2$, i.e. $\int_{\Omega} u_0 u_1 dx > 0$, then we get

$$F'(t) \geq F'(0) = 2(u_0, u_1) + \|D^m u_0\|^2 > \|D^m u_0\|^2.$$

Thus, 3) has been proved.

Step 4: For the case that $E(0) > 0$, we first note that

$$2 \int_0^t (D^m u, D^m u_t) dt = \int_0^t \frac{d}{dt} \|D^m u\|^2 dt = \|D^m u\|^2 - \|D^m u_0\|^2. \tag{4.12}$$

By using Hölder inequality, we have

$$\begin{aligned} \|D^m u\|^2 &= \|D^m u_0\|^2 + \int_0^t (D^m u, D^m u_t) dt \\ &\leq \|D^m u_0\|^2 + \int_0^t \|D^m u\|^2 dt + \int_0^t \|D^m u_t\|^2 dt. \end{aligned} \tag{4.13}$$

So

$$\begin{aligned} F'(t) &= 2(u, u_t) + \|D^m u\|^2 \\ &\leq \|u\|^2 + \|u_t\|^2 + \|D^m u\|^2 \\ &\leq \|u\|^2 + \|u_t\|^2 + \|D^m u_0\|^2 + \int_0^t \|D^m u\|^2 dt + \int_0^t \|D^m u_t\|^2 dt \\ &= F(t) + \|u_t\|^2 + \|D^m u_0\|^2 + \int_0^t \|D^m u_t\|^2 dt. \end{aligned} \tag{4.14}$$

Thus, we have

$$F''(t) - (4 + p)F'(t) + (4 + p)F(t) + K_1 \geq p \int_0^t \|D^m u_t\|^2 dt \geq 0, \tag{4.15}$$

where $K_1 = (4 + 2p)E(0) + (4 + p)\|D^m u_0\|^2$.

Set

$$B(t) = F(t) + \frac{K_1}{4 + p}, \quad t > 0. \tag{4.16}$$

Then $B(t)$ satisfies (2.2). By conditions

$$F'(0) > \left(\frac{4 + p - \sqrt{p^2 + 4p}}{2} \right) \left[F(0) + \frac{(4 + 2p)E(0) + (4 + p)\|D^m u_0\|^2}{4 + p} \right] + \|D^m u_0\|^2$$

and **Lemma 2.2**, then $F'(t) > \|D^m u_0\|^2$ for $t > 0$.

Lemma 4.2 Suppose that $0 \leq p \leq \frac{4m}{n - 2m}$ ($0 \leq p < +\infty$ if $0 \leq n \leq 2m$) and $p \geq 2q$ hold and that either one of the following conditions is satisfied:

- 1) $E(0) < 0$;
- 2) $E(0) = 0$ and $\int_{\Omega} u_0 u_1 dx > 0$;
- 3) $E(0) > 0$ and

$$F'(0) > \left(2 + \frac{p}{2} - \frac{\sqrt{p^2 + 4p}}{2} \right) \left[\left(F(0) + \frac{(4+2p)E(0) + (4+p)\|D^m u_0\|^2}{4+p} \right) \right] + \|D^m u_0\|^2$$

hold.

Then, there exists $t_0 \geq 0$, such that $F'(t) > \|D^m u_0\|^2$ for $t > t_0$.

Proof. By **Lemma 4.1**, $t_0 = t^*$ in case (i) and $t_0 = 0$ in case 2) and 3).

Theorem 4.1 Suppose that $0 \leq p \leq \frac{4m}{n-2m}$ ($0 \leq p < +\infty$ if $0 \leq n \leq 2m$) and

$p \geq 2q$ hold and that eigher one of the following conditions is satisfied:

- 1) $E(0) < 0$;
- 2) $E(0) = 0$ and $\int_{\Omega} u_0 u_1 dx > 0$;
- 3) $0 < E(0) < \frac{(F'(t_0) - \|D^m u_0\|^2)^2}{8[F(t_0) + (T_1 - t_0)\|D^m u_0\|^2]}$ and

$$F'(0) > \left(\frac{4+p-\sqrt{p^2+4p}}{2} \right) \left[\left(F(0) + \frac{(4+2p)E(0) + (4+p)\|D^m u_0\|^2}{4+p} \right) \right] + \|D^m u_0\|^2$$

hold.

Then the solution u blow up at finite T^* . And T^* can be estimated by (4.26)-(4.29), respectively, according to the sign of $E(0)$.

Proof. Let

$$J(t) = \left(F(t) + (T_1 - t)\|D^m u_0\|^2 \right)^{\frac{p}{4}}, \quad t \in [0, T_1], \tag{4.17}$$

where T_1 is some certain constant which will be chosen later. Then we get

$$J'(t) = -\frac{p}{4} J(t)^{1+\frac{4}{p}} \left(F'(t) - \|D^m u_0\|^2 \right), \tag{4.18}$$

and

$$J''(t) = -\frac{p}{4} J(t)^{1+\frac{8}{p}} V(t), \tag{4.19}$$

where $V(t) = F''(t) \left(F(t) + (T_1 - t)\|D^m u_0\|^2 \right) - \frac{4+p}{4} \left(F'(t) - \|D^m u_0\|^2 \right)^2$.

By the Hölder inequality, we obtain

$$\begin{aligned} F'(t) &= 2(u, u_t) + \|D^m u\|^2 \\ &= 2(u, u_t) + \|D^m u_0\|^2 + 2\int_0^t (D^m u, D^m u_t) dt \\ &\leq 2\|u\| \cdot \|u_t\| + \|D^m u_0\|^2 + 2\int_0^t \|D^m u\|^2 dt \cdot \int_0^t \|D^m u_t\|^2 dt \\ &= \|D^m u_0\|^2 + 2(\sqrt{PR} + \sqrt{QS}), \end{aligned} \tag{4.20}$$

where $P = \|u(t)\|^2$, $Q = \int_0^t \|u(s)\|^2 ds$, $R = \|u_t(t)\|^2$, $S = \int_0^t \|u_t(s)\|^2 ds$.

By 1) of **Lemma 4.1**, we get

$$F''(t) \geq -(4+2p)E(0) + (4+p)(R+S). \tag{4.21}$$

Then, we obtain

$$\begin{aligned} V(t) &\geq [-(4+2p)E(0) + (4+p)(R+S)] \left(F(t) + (T_1 - t) \|D^m u_0\|^2 \right) \\ &\quad - (4+p) (\sqrt{PR} + \sqrt{QS})^2 \\ &= -(4+2p)E(0)J(t)^{-\frac{4}{p}} + (4+p)(R+S)(T_1 - t) \|D^m u_0\|^2 \\ &\quad + (4+p) \left[(R+S)(P+Q) - (\sqrt{PR} + \sqrt{QS})^2 \right] \\ &\geq -(4+2p)E(0)J(t)^{-\frac{4}{p}}, t \geq t_0. \end{aligned} \tag{4.22}$$

Therefore, we get

$$J''(t) \leq \left(p + \frac{p^2}{2} \right) E(0) J(t)^{\frac{p+4}{p}}, t \geq t_0. \tag{4.23}$$

Note that by **Lemma 4.2**, $J'(t) < 0, t > t_0$. Multiplying (4.23) by $J'(t)$ and integrating it from t_0 to t , we have

$$J'(t)^2 \geq \alpha + \beta J(t)^{\frac{2p+4}{p}}, t \geq t_0, \tag{4.24}$$

where $\alpha = \frac{p^2}{16} J(t_0)^{\frac{2p+4}{p}} \left[\left(F'(t_0) - \|D^m u_0\|^2 \right)^2 - 8E(0)J(t_0)^{\frac{4}{p}} \right]$, and

$$\beta = \frac{p^2}{2} E(0).$$

When $E(0) < 0$ and $E(0) = 0$, we obviously have $\alpha > 0$. When $E(0) > 0$,

we also have $\alpha > 0$ by condition $E(0) < \frac{\left(F'(t_0) - \|D^m u_0\|^2 \right)^2}{8 \left[F(t_0) + (T_1 - t_0) \|D^m u_0\|^2 \right]}$.

Then by **Lemma 2.3**, there exists a finite time T^* such that $\lim_{t \rightarrow T^{*-}} J(t) = 0$ and the upper bounds of T^* are estimated respectively according to the sign of $E(0)$. This will imply that

$$\lim_{t \rightarrow T^{*-}} \left(\|u(t)\|^2 + \int_0^t \|D^m u(s)\|^2 ds \right) = +\infty. \tag{4.25}$$

Next, T^* are estimated respectively according to the sign of $E(0)$ and **Lemma 2.3**.

In case 1), we have

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}. \tag{4.26}$$

Furthermore, if $J(t_0) < \min \left\{ 1, \sqrt{\frac{\alpha}{-\beta}} \right\}$, then we have

$$T^* \leq t_0 + \sqrt{\frac{1}{-\beta}} \ln \frac{\sqrt{\frac{\alpha}{-\beta}}}{\sqrt{\frac{\alpha}{-\beta}} - J(t_0)}. \quad (4.27)$$

In case 2), we get

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)} \quad \text{or} \quad T^* \leq t_0 - \frac{J(t_0)}{\sqrt{\alpha}}. \quad (4.28)$$

In case 3), we obtain

$$T^* \leq \frac{J(t_0)}{\sqrt{\alpha}} \quad \text{or} \quad T^* \leq t_0 + 2^{\frac{3p+4}{2p}} \frac{pc}{4\sqrt{\alpha}} \left\{ 1 - [1 + cJ(t_0)]^{\frac{2}{p}} \right\}, \quad (4.29)$$

where $c = \left(\frac{\beta}{\alpha}\right)^{\frac{p}{p+8}}$. Note that in case 1), $t_0 = t^*$ is given **Lemma 4.1**, and in case 2) and case 3) $t_0 = 0$.

Remark 4.1 [10] The choice of T_1 in (4.17) is possible under some conditions.

1) In the case $E(0) = 0$, we can choose $T_1 \geq \frac{16\|u_0\|^2}{p^2 K^2 \|u_1\|^2}$. In particular, we

choose $T_1 = \frac{16\|u_0\|^2}{p^2 K^2 \|u_1\|^2}$, then we get $T^* \leq \frac{16\|u_0\|^2}{p^2 K^2 \|u_1\|^2}$.

2) In the case $E(0) < 0$, we can choose T_1 as in 1) if $\int_{\Omega} u_0 u_1 dx > 0$ or $T_1 \geq t^* - \frac{J(t_0)}{J'(t_0)}$ if $\int_{\Omega} u_0 u_1 dx \leq 0$.

3) For the case $E(0) > 0$. Under the condition $E(0) < \min\{k_1, k_2\}$,

$$\text{here } k_1 = \frac{(4+p) \left[F'(0) - \frac{4+p-\sqrt{p^2+4p}}{2} F(0) - \frac{6+p-\sqrt{p^2+4p}}{2} \|D^m u_0\|^2 \right]}{(4+p-\sqrt{p^2+4p})(2+p)},$$

$$k_2 = \frac{4 \left(\int_{\Omega} u_0 u_1 dx \right)^2 - 1}{8p \|D^m u_0\|^2} \left[p - 4 \|D^m u_0\|^2 \right],$$

if $\|D^m u_0\|^2 < \frac{p}{4}$, T_1 is chosen to satisfy $k_3 \leq T_1 \leq k_4$, where $k_3 = \frac{4\|u_0\|^2}{p-4\|D^m u_0\|^2}$,

$$k_4 = \frac{4 \left(\int_{\Omega} u_0 u_1 dx \right)^2 - 8E(0) \|u_0\|^2 - 1}{8E(0) \|D^m u_0\|^2}. \quad \text{Therefore, we have}$$

$$T \leq T^* \leq \frac{k_3}{\sqrt{4 \left(\int_{\Omega} u_0 u_1 dx \right)^2 - 8E(0) k_3}}.$$

5. Conclusion

In this paper, we prove that nonlinear wave equations of higher-order Kirchhoff Type with Strong Dissipation exist unique local solution on

$u \in C^0([0, T]; H^{2m}(\Omega) \cap H_0^m(\Omega))$, $u_t \in C^0([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^m(\Omega))$. Then, we establish three blow-up results for certain solutions in the case 1): $E(0) < 0$, in the case 2): $E(0) = 0$ and in the case 3): $E(0) > 0$. At last, we consider that the estimation of the upper bounds of the blow-up time T^{*-} is given for different initial energy.

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