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The AK Transform

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Abstract

In this brief communication we present a new integral transform, so far unknown, which is applicable, for instance, to studying the kinetic theory of natural eigenmodes or transport excited in plasmas with bounded distribution functions such as in Q machines/plasma diodes or in the scrap-off layer of Tokamak fusion plasmas. The results are valid for functions of $L^p\{\mathbb{R},\sigma_S,\mu\}$ function spaces—Lebesgue spaces, which are defined using a natural generalization of the *p*-norm for finite-dimensional vector spaces, where $\mathbb{R}$ is the real set, σ_S is the σ-algebra of Lebesgue measurable sets, and μ the Lebesgue measure. $AK: L^p[0,L]\to L^p[0,L]$, so that $f\to AK(f)$. Note that, using a simpler notation, more natural/known to engineers, *f* could be considered any piecewise continuous function, that is: $f\in PC[0,L]$. Here $PC[0,L]$ is a Euclidian space with the usual norm (inner product: $\langle f,f\rangle$) given by: $\|f\|^2=\langle f,f\rangle=\int_0^L[f(x)]^2\mathrm{d}x$ [1].

Keywords

Integral Transform, Lebesgue Measures, Kinetic Theory of Bounded Plasmas, Natural Eigenmodes, Transport, Q-Machines, Plasma Diodes, Tokamak, Nuclear Fusion

1. Prolegomenon

The mathematics that has hatched the novel operator presented in this paper is connected to the integral equation that arises from the technique to obtain eigenmodes of bounded plasmas, with the caveat of one-dimensional geometry and no particles collision, that is collisionless plasma, however coupled to an external electric circuit via the electrodes, a model which is very useful to describe the external control of plasma dynamics. In order to get the mathematics and hence the operator under consideration, an integral-equation method was developed for solving a general linearized perturbation problem for an one-dimensional, uniform and collisionless plasma with thin sheaths, bounded by two planar electrodes [2] [3] [4] [5] [6]. The operator is the result of a natural combination of the basic equations involved in this technique. The underlying system of integral equations related to this physical system consists of (1) the collisionless Vlasov plasma kinetic equations for all particles species involved (electrons and/or ions), (2) Poisson's equation, (3) the equation of total current conservation, (4) the particles (distribution-function) boundary conditions at the left- and right–hand electrodes, fed by the plasma and the external-current system, and (5) the external circuit condition.

The method allows for very general equilibrium, boundary, and external circuit conditions. Using Laplace's transformation in both time and space, it is set up to handle the complete linearized initial-value problem and also yields the solution to the eigenmode problem as a by-product, using the Nyquist technique in this case which is a very useful method to access the zeros of a complex function $f(\omega)$, considering $f(\omega)$ as a mapping of the complex ω plane. For instance, this method is also applicable to studying the Pierce diode with non-trivial external circuit dynamics, and it is also useful to studying the important problem of plasma-wall interactions in Tokamaks, which can induce undesirable plasma cooling leading in many cases to termination of the Tokamak discharge and, hence of nuclear-fusion reactions. This problem is of paramount importance for present-day fusions machines such as JET—Joint European Torus and future ones, such as ITER and DEMO.

Genesis of the AK Transform

Here, to obtain the AK transform, it is sufficient to use the results of the special case of longitudinal oscillations in the negatively biased single-ended Q machine.

As already mentioned, the mathematical operator under consideration, takes shape just when we combine the Laplace-transformed integral-equation system for the perturbed plasma system variables—electric field, electric current, distribution functions.

After combining the basic equations, the Laplace component of the perturbed electric-field solution of the posed problem can be written as

$$E(x,\omega) = B_{op}^{-1}\left[x,[x'],\omega\right]\left\{k_5\left([x,],\omega\right) j_e\left([x,],\omega\right)\right\} + \left\{k_8\left([x,],\omega\right)\right\}$$

with $B_{op}\left[x,[x'],\omega\right]$ an operator acting on x' and defined as

$$B_{op}\left[x,[x'],\omega\right]\{E\} = \left\{I_{op}\left[x,[x']\right] + J_{op}\left[x,[x'],\omega\right]\right\} E(x,\omega)$$

where I_{op} is the identity integral operator (Dirac delta) and J_{op}, k_5, k_8, j_e, a, L, D, and all the remaining terms are defined in references [2] [3] [4] [5] [6] and need not be displayed/redefined here. Note that L and D are related to the physical dimension of the plasma machine and J_{op} is a complicated operator defined in [2] [3] [4] [5] [6], which, however, after some algebra assumes a simple form, so that we can write

$$B_{op}\left[x,\left[x'\right],\omega\right]\{\cdot\}=\left[I_{op}\{\cdot\}+a\left(x,\omega\right)\int_0^L\{\cdot\}\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\right]$$

where $a\left(x,\omega\right)$ is a function that depends on the physical problem under consideration (type of machine and external circuit and type of plasma geometry and physical properties).

We can write the operator $B_{op}\left(x,\omega,y\right)$ (where y, the dummy variable-integration variable, is in place of the original integration variable $\left[x'\right]$) in a "more mathematical" way, replacing the real plasma physical system operator for the operator seed of the AK transform, and also replacing, later, the electric field by an arbitrary more mathematical function f, thus

$$AK\left(E\right)=E\left(x,\omega\right)+a\left(x,\omega\right)\int_0^L E\left(y,\omega\right)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\equiv F\left(x,\omega,L,\ D\right)$$

In order to proceed, we work a bit further the concept of *L*– space, which will be useful to where we intend to move:

The space $L^p\left[0,L\right]$ is also called the space of p^{th} —power integrable functions, where $0<p<+\infty,\int\left\|f\right\|^p\mathrm{d}\mu<+\infty,\quad\mu$ the Lebesgue measure.

We consider here, however, a much simpler way to start with our conjecture. Let us then assume that f is a piecewise continuous function Euclidian space $PC\left[0,\ L\right]$, where L is a real number. Of course, the conjecture is also valid for a more formal space as $L^p\left[\mathbb{R},\sigma_S,\mu\right]$.

Therefore, we finally define our novel transform, namely the **AK** transform, as below, where here we have replaced, as mentioned above, the electric field with an arbitrary function $f\left(x\right)$. Note that we leave **ω** out of this notation just for the sake of simplicity as AK operates only in functions from spaces defined by the domain of the variable x. The result of the operation of the transform AK on f is given by

$$AK\left(f\right)=f\left(x\right)+a\left(x\right)\int_0^L f\left(y\right)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\equiv F\left(x,\ L,\ D\right)\tag{1}$$

$$AK\left(f\right)\equiv\left(I+S\right)\left(f\right)\equiv F\left(f\left(x\right),a\left(x\right),L,D\right),$$

where I is the identity operator and

$$S\left(f\right)=a\left(x\right)\int_0^L f\left(y\right)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y=C\left(L,D\right)a\left(x\right).$$

In this way, we can write the operator AK as

$$AK\equiv\left(I+S\right).$$

where the integral operator S is given by

$$S\left(*\right)=a((x)\int_0^L(*)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y.$$

Here $a\left(x\right)$ is a given/known continuous function or piecewise continuous function, at least, and L and D are free given/known positive real parameters.

We can also write

$$S(f) \equiv (AK - I)(f) = a(x)C(L,D),$$

$$AK(f) = f(x) + a(x)C(L,D).$$

We can show that AK is invertible, it is linear, it is injective, and it is limited. The inversion can be shown as below:

$$AK^{-1}(F) \equiv \left(I - \frac{S}{1 + A(L,D)} \right)(F) = f(x),$$

in a way that $AK^{-1}AK \equiv AK^{-1}AK \equiv I$, where I is the identity operator and $S \equiv (AK - I)$, L, D are positive real numbers, f is a continuous or piecewise continuous function and $a(x)$ is a prescribed/given continuous or piecewise continuous function as well.

2. Inversion

Applying now AK to Equation (1) from the left-handside we have

$$\begin{aligned} AKAK(f) &\equiv AK(f(x)) + AK\left(a(x)\int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{dy} \right) \\ &\equiv AK(F(x,L,D)) \end{aligned}$$

$$\begin{aligned} AKAK(f) &= AK(f(x)) + C(L,D))AK(a(x)) \\ &= F(x,\ L,\ D)\ + C(L,D)AK(a(x)) \end{aligned}$$

Thus,

$$AKAK(f) = f(x) + (2 + A(L,D))C(L,D)a(x) \tag{2}$$

where,

$$A(L,D) = \int_0^L a(y)\sin\left(\frac{L-y}{D}\right)\mathrm{dy}.$$

Applying now AK^{-1} from the left-hand side on both Equations (1) and (2), we, therefore, find again the original function $f(x)$,

$$f(x) = AK^{-1}(f) + C(L,D)AK^{-1}(a(x)) \tag{1'}$$

and,

$$AK(f) \equiv AK^{-1}(f) + (2 + A(L,D))C(L,D)AK^{-1}(a(x)) = F(x,\ L,D) \tag{2'}$$

We then have a system of two equations for

$$AK^{-1}(f) \text{ and } C(L,D)AK^{-1}(a(x)).$$

This implies that, from (1'),

$$AK^{-1}(f) = f(x) - C(L,D)AK^{-1}(a(x)),$$

and from (2'),

$$AK^{-1}(f) = AK(f(x)) - (2 + A(L,D))C(L,D)AK^{-1}(a(x)).$$

Therefore,

$$\begin{aligned} AK^{-1}(f) &= f(x) - C(L,D)AK^{-1}(a(x)) \\ &= AK(f) - (2 + A((L,D))C(L,D)AK^{-1}(a(x)) \\ &= F(x,L,D) - (2 + A(L,D))C(L,D)AK^{-1}(a(x)). \end{aligned}$$

Thus,

$$\begin{aligned} C(L,D)AK^{-1}(a(x)) &= \frac{(AK - I)}{(1 + A(L,D))}(f(x)) \\ &\equiv \frac{S}{(1 + A(L,D))}(f(x)) \\ &= \frac{F(x,L,D) - f(x)}{(1 + A(L,D))}. \end{aligned}$$

In conclusion,

$$\begin{aligned} AK^{-1}(f) &= f(x) - C(L,D)AK^{-1}(a(x)) \\ &= f(x) - \frac{S}{(1 + A(L,D))}(f) \\ &= \left(I - \frac{S}{(1 + A(L,D))} \right)(f) \end{aligned}$$

$$\begin{aligned} AK^{-1}(f) &= f(x) - C(L,D)AK^{-1}(a(x)) \\ &= f(x) - \frac{S}{(1 + A(L,D))}(f) \\ &= \left(I - \frac{S}{(1 + A(L,D))} \right)(f) \\ &= f(x) + \frac{F(x,L,D) - f(x)}{(1 + A(L,D))} \\ &= \left(1 - \frac{1}{(1 + A(L,D))} \right) f(x) + \frac{F(x,L,D)}{(1 + A(L,D))} \end{aligned}$$

and so

$$AK^{-1}(f) = (I - \frac{S}{(1 + A(L,D))})(f),$$

$$AK^{-1} \equiv \left(I - \frac{S}{1 + A(L,D)} \right),$$

$$AK^{-1} \equiv \left(I - \frac{S}{1 + A(L,D)} \right).$$

Therefore,

$$AK^{-1}(F) \equiv \left(I - \frac{S}{1 + A(L,D)} \right)(F) = f(x)$$

In fact,

$$AKAK^{-1}(f) = AK\left(I - \frac{S}{(1+A(L,D))}\right)(f)$$

But,

$$AK \equiv (I+S).$$

Thus,

$$\begin{aligned}AKAK^{-1}(f) &= AK\left(I - \frac{S}{(1+A(L,D))}\right)(f)\\ &= (I+S)\left(I - \frac{S}{(1+A(L,D))}\right)(f)\\ &= (I+S)\left(f(x) - \frac{a(x)C(L,D)}{1+A(L,D)}\right)\\ &= f(x) - \frac{a(x)C(L,D)}{1+A(L,D)} + S(f) - \frac{C(L,D)S(a(x))}{1+A(L,D)}\\ &= f(x)\end{aligned}$$

So, indeed,

$$AKAK^{-1}(f) = f(x)$$

Now consider,

$$AK^{-1}AK(f) \equiv AK^{-1}(F) \equiv AK^{-1}(f) + C(L,D)AK^{-1}(a(x))$$

But,

$$C(L,D)AK^{-1}(a(x)) = \frac{S}{(1+A(L,D))}(f(x))$$

and

$$AK^{-1}(f) = f(x) - \frac{S}{(1+A(L,D))}(f)$$

Thus,

$$\begin{aligned}AK^{-1}AK(f) &= AK^{-1}(f) + C(L,D)AK^{-1}(a(x))\\ &= f(x) - \frac{S}{(1+A(L,D))}(f) + \frac{S}{(1+A(L,D))}(f)\\ &= f(x)\end{aligned}$$

We have then,

$$AK^{-1}AK(f) \equiv AK^{-1}(F) = f(x).$$

Therefore, we have completed the two ways inversion.

The AK's transform can be extended to square integrable functions, that is:

$$AK(f) = f(x) + a(x)\int_{-\infty}^{\infty} f(y)\exp\left(\pm j\omega\left(\frac{L-y}{D}\right)\right)\mathrm{d}y = F(x,\ D)$$

Moving further, we will show that:

3. Properties

Moving further, we will show some fundamental properties of AK.

3.1. Linearity

In fact, consider $AK : L^p(0,L) \to L^p(0,L)$, so that $f \to AK(f)$ where

$$(AKf)(x) = f(x) + a(x)\int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y$$

Thus,

$$AK(\alpha f + \mu g)(x) = (\alpha f + \mu g)(x) + a(x)\int_0^L (\alpha f + \mu g)(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y$$

$$AK(\alpha f + \mu g)(x) = \alpha\left(f(x) + \int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\right)$$
$$+\mu\left(g(x) + \int_0^L g(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\right)$$

3.2. AK Is Injective

Since AK is linear, we have only to show that $AK(f) = 0 \Rightarrow f = 0$.

Indeed,

$$AK(f) = 0 = f(x) + a(x)\int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y$$

$$\|f(x)\| = \|a(x)\|\left[\int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\right]$$

Thus,

$$\|f(x)\| \le \|a(x)\|\int_0^L\left\|f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y\right\|$$

$$\|f(x)\| \le \|a(x)\|\int_0^L\|f(y)\|\mathrm{d}y$$

if we assume $p = 1$ (*i.e.* city block distance/Manhattan distance).

If we use now the inequality $\left[1-\int_0^L\|a(x)\|\mathrm{d}x\right]\int_0^L\|f(x)\|\mathrm{d}x \le 0,$ (note that we are free to choose this ansatz, since we can specific $a(x)$ in a way that $\int_0^L\|a(x)\|\mathrm{d}x < 1.$) we have $\left[1-\int_0^L\|a(x)\|\mathrm{d}x\right] > 0$, since $\int_0^L\|a(x)\|\mathrm{d}x < 1$.

Thus, we are forced to conclude that

$$\int_0^L\|f(x)\|\mathrm{d}x \le 0 \Rightarrow f = 0.$$

if $p > 1$.

Here we can also show that if

$$\int_0^L 1\|f(y)\|\mathrm{d}y \le \left[\int_0^L 1^q\mathrm{d}y\right]^{\frac{1}{q}}\left[\int_0^L\|f(y)\|^{\frac{1}{p}}\mathrm{d}y\right] \le L^{\frac{p-1}{p}}\|f\|_{L^p(0,L)}$$

$$\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow q=\frac{p}{p-1}$$

Also,

$$\|f\|_{L^p(0,L)} \le \|a\|_{L^p(0,L)} < 1 \Rightarrow \|f\|_{L^p(0,L)} = 0 \Rightarrow f = 0!$$

3.3. AK Is Limited

We show here that AK is limited (*i.e.* continuous).

Indeed,

$$\begin{aligned}\|AK(f)\|_{L^p(0,L)} &= \left\| f(x)+a(x)\int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y \right\|_{L^p(0,L)} \\ &\le \|f\|_{L^p(0,L)} + \|a\|_{L^p(0,L)} \left\| \int_0^L f(y)\sin\left(\frac{L-y}{D}\right)\mathrm{d}y \right\| \\ &\le \|f\|_{L^p(0,L)} + \|a\|_{L^p(0,L)} \int_0^L \|f(y)\|\mathrm{d}y \\ &\le \|f\|_{L^p(0,L)} + \|a\|_{L^p(0,L)} L^{\frac{p-1}{p}} \|f\|_{L^p(0,L)} \\ &\le \left(1+\|a\|_{L^p(0,L)} L^{\frac{p-1}{p}}\right)\|f\|_{L^p(0,L)}\end{aligned}$$

and so it is limited.

4. Conclusions

In short, a new integral operator AK has been created which is useful to studying kinetic theory for bounded plasmas, a subject of paramount importance for nuclear fusion applications (mainly to study plasma wall interactions), as well as all types of technological plasma applications.

From the mathematical point of view, much can still be done in studying this operator to fully understand its mathematical structure and applicability to solving problems involving, for instance, differential, integral, and integrodifferential equations.

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