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The Generalized *r*-Whitney Numbers

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Abstract

In this paper, we define the generalized *r*-Whitney numbers of the first and second kind. Moreover, we drive the generalized Whitney numbers of the first and second kind. The recurrence relations and the generating functions of these numbers are derived. The relations between these numbers and generalized Stirling numbers of the first and second kind are deduced. Furthermore, some special cases are given. Finally, matrix representation of the relations between Whitney and Stirling numbers is given.

Keywords

Whitney Numbers, *r*-Whitney Numbers, *p*-Stirling Numbers, Generalized *q*-Stirling Numbers, Generalized Stirling Numbers

1. Introduction

The *r*-Whitney numbers of the first and second kind were introduced, respectively, by Mezö [1] as

$$m^n(x)_n = \sum_{k=0}^{n} w_{m,r}(n,k)(mx+r)^k, \tag{1}$$

$$(mx+r)^n = \sum_{k=0}^{n} W_{m,r}(n,k)m^k(x)_k. \tag{2}$$

Many properties of these numbers and their combinatorial interpretations can be seen in Mezö [2] and Cheon [3]. At $r=1$ the *r*-Whitney numbers are reduced to the Whitney numbers of Dowling lattice introduced by Dowling [4] and Benoumhani [5].

In this paper we use the following notations (see [6] [7] [8]):

Let $\bar{\boldsymbol{\alpha}} = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ where $\alpha_i, i = 0,1,\cdots,n-1$ are real numbers.

$$(\alpha_i)_n = \prod_{j=0,\, j\neq i}^{n-1}(\alpha_i - \alpha_j), (\alpha_i)_0 = 1 \text{ and } (x;\bar{\boldsymbol{\alpha}})_n = \prod_{i=0}^{n-1}(x-\alpha_i), (x;\bar{\boldsymbol{\alpha}})_0 = 1. \tag{3}$$

$$[x;\bar{\boldsymbol{\alpha}}]_{\underline{n},q} = \prod_{i=0}^{n-1}[x-\alpha_i]_q = q^{-\sum_{i=0}^{n-1}\alpha_i}\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_n, \tag{4}$$

where $\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_n = \left([x]_q-[\alpha_0]_q\right)\left([x]_q-[\alpha_1]_q\right)\cdots\left([x]_q-[\alpha_{n-1}]_q\right)$,

$[x]_q = \dfrac{1-q^x}{1-q}$.

This paper is organized as follows:

In Sections 2 and 3 we derive the generalized *r*-Whitney numbers of the first and second kind.The recurrence relations and the generating functions of these numbers are derived. Furthermore, some interesting special cases of these numbers are given. In Section 4 we obtain the generalized Whitney numbers of the first and second kind by setting $r=1$. We investigate some relations between the generalized *r*-Whitney numbers and Stirling numbers and generalized harmonic numbers in Section 5. Finally, we obtain a matrix representation for these relations in Section 6.

2. The Generalized *r*-Whitney Numbers of the First Kind

Definition 1. *The generalized r*-Whitney *numbers of the first kind* $w_{m,r;\bar{\alpha}}(n,k)$ *with parameter*

$\bar{\boldsymbol{\alpha}} = (\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$ are defined by

$$m^n(x;\bar{\boldsymbol{\alpha}})_n = \sum_{k=0}^{n} w_{m,r;\bar{\alpha}}(n,k)(mx+r)^k, \tag{5}$$

where $w_{m,r;\bar{\alpha}}(0,0) = w_{m,r;\bar{\alpha}}(n,n) = 1$ and $w_{m,r;\bar{\alpha}}(n,k) = 0$ for $k>n$.

Theorem 2. *The generalized r*-Whitney *numbers of the first kind* $w_{m,r;\bar{\alpha}}(n,k)$ *satisfy the recurrence relation*

$$w_{m,r;\bar{\alpha}}(n,k) = w_{m,r;\bar{\alpha}}(n-1,k-1) - (r+m\alpha_{n-1})w_{m,r;\bar{\alpha}}(n-1,k), \tag{6}$$

for $n \ge k \ge 1$ and $w_{m,r;\bar{\alpha}}(n,0) = (-1)^n\prod_{i=1}^{n}(r+m\alpha_{i-1})$.

Proof. Since $m^n(x;\bar{\boldsymbol{\alpha}})_n = (mx+r-r-m\alpha_{n-1})m^{n-1}(x;\bar{\boldsymbol{\alpha}})_{n-1}$, we have

$$\begin{aligned}\sum_{k=0}^{n} w_{m,r;\bar{\alpha}}(n,k)(mx+r)^k &= \sum_{k=0}^{n-1} w_{m,r;\bar{\alpha}}(n-1,k)(mx+r)^{k+1}\\ &\quad -(r+m\alpha_{n-1})\sum_{k=0}^{n-1} w_{m,r;\bar{\alpha}}(n-1,k)(mx+r)^k\\ &= \sum_{k=1}^{n} w_{m,r;\bar{\alpha}}(n-1,k-1)(mx+r)^k\\ &\quad -(r+m\alpha_{n-1})\sum_{k=0}^{n-1} w_{m,r;\bar{\alpha}}(n-1,k)(mx+r)^k.\end{aligned}$$

Equating the coefficients of $(mx+r)^k$ on both sides, we get Equation (6).

Using Equation (6) it is easy to prove that

$w_{m,r;\bar{\alpha}}(n,0) = (-1)^n\prod_{i=1}^{n}(r+m\alpha_{i-1})$.. □

Special cases:

1. Setting $\alpha_i = 0$, for $i = 0,1,\cdots,n-1$, hence Equation (5) is reduced to

$$m^n x^n = \sum_{k=0}^{n} w_{m,r;\mathbf{0}}(n,k)(mx+r)^k. \tag{7}$$

Thus

$$m^n x^n = \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} r^{k-i} w_{m,r;\mathbf{0}}(n,k)\right)(mx)^i, \tag{8}$$

hence

$$\sum_{k=i}^{n}\binom{k}{i} r^{k-i} w_{m,r;\mathbf{0}}(n,k) = \delta_{ni}, \tag{9}$$

where $\mathbf{0} = (0,0,\cdots,0)$ and δ_{ni} is Kronecker's delta.

2. Setting $\alpha_i = \alpha$, for $i = 0,1,\cdots,n-1$, hence Equation (5) is reduced to

$$m^n (x-\alpha)^n = \sum_{k=0}^{n} w_{m,r;\boldsymbol{\alpha}}(n,k)(mx+r)^k, \tag{10}$$

therefore we have

$$m^n \sum_{i=0}^{n}\binom{n}{i} x^i (-\alpha)^{n-i} = \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} r^{k-i} w_{m,r;\boldsymbol{\alpha}}(n,k)\right) m^i x^i. \tag{11}$$

Equating the coefficient of x^i on both sides, we get

$$m^{n-i}\binom{n}{i}(-\alpha)^{n-i} = \sum_{k=i}^{n}\binom{k}{i} r^{k-i} w_{m,r;\boldsymbol{\alpha}}(n,k), \tag{12}$$

where $\boldsymbol{\alpha} = (\alpha,\alpha,\cdots,\alpha)$.

3. Setting $\alpha_i = i$, for $i = 0,1,\cdots,n-1$, hence Equation (5) is reduced to

$$m^n (x)_n = \sum_{k=0}^{n} w_{m,r;\boldsymbol{i}}(n,k)(mx+r)^k, \tag{13}$$

where $\boldsymbol{i} = (0,1,\cdots,n-1)$ and $w_{m,r;\boldsymbol{i}}(n,k) = w_{m,r}(n,k)$ are the *r*-Whitney numbers of the first kind.

4. Setting $\alpha_i = i$, for $i = 0,1,\cdots,n-1$, and $r = -a$ hence $w_{m,-a;\boldsymbol{\alpha}}(n,k)$ are the noncentral Whitney numbers of the first kind, see [9].

5. Setting $\alpha_i = -i\alpha$, for $i = 0,1,\cdots,n-1$, $r = 0$ and $m = 1$, hence Equation (5) is reduced to

$$(x;-\alpha)_n = \sum_{k=0}^{n} w_{1,0;-\boldsymbol{i\alpha}}(n,k)(x)^k, \tag{14}$$

where $-\boldsymbol{i\alpha} = (0,-\alpha,\cdots,-(n-1)\alpha)$ and $w_{1,0;-\boldsymbol{i\alpha}}(n,k) = \begin{bmatrix} n \\ k \end{bmatrix}^{(\alpha)}$ are the translated Whitney numbers of the first kind defined by Belbachir and Bousbaa [10].

6. Setting $\alpha_i = i^p$, for $i = 0,1,\cdots,n-1$, hence Equation (5) is reduced to

$$m^n \prod_{i=0}^{n}(x-i^p) = \sum_{k=0}^{n} w_{m,r;\boldsymbol{i}^p}(n,k)(mx+r)^k. \tag{15}$$

Sun [11] defined p-Stirling numbers of the first kind as

$$\prod_{i=0}^{n}(x-i^p) = \sum_{i=0}^{n} s_1(n,i,p)\, x^i,$$

therefore, we have

$$m^n\sum_{i=0}^{n}s_1(n,i,p)x^i=\sum_{k=0}^{n}w_{m,r;\boldsymbol{i}^p}(n,k)(mx+r)^k$$
$$=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}r^{k-i}w_{m,r;\boldsymbol{i}^p}(n,k)\right)(mx)^i.$$

Equating the coefficient of x^i on both sides, we get

$$s_1(n,i,p)=\frac{1}{m^{n-i}}\sum_{k=i}^{n}\binom{k}{i}r^{k-i}w_{m,r;\boldsymbol{i}^p}(n,k),\tag{16}$$

where $\boldsymbol{i}^p=\left(0^p,1^p,\cdots,(n-1)^p\right)$.

7. Setting $\alpha_i=[\alpha_i]_q$, and $x=[x]_q$ for $i=0,1,\cdots,n-1$, Equation (5) is reduced to

$$m^n\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_n=\sum_{k=0}^{n}w_{m,r;[\bar{\alpha}]_q}(n,k)\left(m[x]_q+r\right)^k,\tag{17}$$

where $[\bar{\boldsymbol{\alpha}}]_q=\left([\alpha_0]_q,[\alpha_1]_q,\cdots,[\alpha_{n-1}]_q\right)$.

El-Desouky and Gomaa [12] defined the generalized q-Stirling numbers of the first kind by

$$[x;\bar{\boldsymbol{\alpha}}]_{\underline{n},q}=q^{-\sum_{i=0}^{n-1}\alpha_i}\sum_{k=0}^{n}s_{q,\bar{\alpha}}(n,k)[x]_q^k,\tag{18}$$

hence, we get

$$\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_n=\sum_{k=0}^{n}s_{q,\bar{\alpha}}(n,k)[x]_q^k,$$

thus we have

$$m^n\sum_{i=0}^{n}s_{q,\bar{\alpha}}(n,i)[x]_q^i=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}r^{k-i}w_{m,r;[\bar{\alpha}]_q}(n,k)\right)m^i[x]_q^i.\tag{19}$$

Equating the coefficient of $[x]_q^i$ on both sides, we get

$$s_{q,\bar{\alpha}}(n,i)=\frac{1}{m^{n-i}}\sum_{k=i}^{n}\binom{k}{i}r^{k-i}w_{m,r;[\bar{\alpha}]_q}(n,k).\tag{20}$$

3. The Generalized *r*-Whitney Numbers of the Second Kind

Definition 3. *The generalized r*-Whitney *numbers of the second kind* $W_{m,r}(n,k;\bar{\boldsymbol{\alpha}})$ *with parameter* $\bar{\boldsymbol{\alpha}}=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$ are defined by

$$(mx+r)^n=\sum_{k=0}^{n}W_{m,r;\bar{\alpha}}(n,k)m^k(x;\bar{\boldsymbol{\alpha}})_k,\tag{21}$$

where $W_{m,r;\bar{\alpha}}(0,0)=W_{m,r;\bar{\alpha}}(n,n)=1$ and $W_{m,r;\bar{\alpha}}(n,k)=0$ for $k>n$.

Theorem 4. *The generalized r*-Whitney *numbers of the second kind* $W_{m,r;\bar{\alpha}}(n,k)$ *satisfy the recurrence relation*

$$W_{m,r;\bar{\alpha}}(n,k)=W_{m,r;\bar{\alpha}}(n-1,k-1)+(r+m\alpha_k)W_{m,r;\bar{\alpha}}(n-1,k)\tag{22}$$

for $n\ge k\ge 1$, and $W_{m,r;\bar{\alpha}}(n,0)=(r+m\alpha_0)^n$.

Proof. Since $(mx+r)^n=(mx-m\alpha_k+m\alpha_k+r)(mx+r)^{n-1}$, we have

$$\sum_{k=0}^{n} W_{m,r;\bar{\alpha}}(n,k) m^k (x;\bar{\alpha})_k = \sum_{k=0}^{n-1} W_{m,r;\bar{\alpha}}(n-1,k) m^{k+1} (x;\bar{\alpha})_{k+1}$$
$$+ (r + m\alpha_k) \sum_{k=0}^{n-1} W_{m,r;\bar{\alpha}}(n-1,k) m^k (x;\bar{\alpha})_k$$
$$= \sum_{k=1}^{n} W_{m,r;\bar{\alpha}}(n-1,k-1) m^k (x;\bar{\alpha})_k$$
$$+ (r + m\alpha_k) \sum_{k=0}^{n-1} W_{m,r;\bar{\alpha}}(n-1,k) m^k (x;\bar{\alpha})_k .$$

Equating the coefficient of $(x;\bar{\alpha})_k$ on both sides, we get Equation (22).

From Equation (22) it is easy to prove that $W_{m,r;\bar{\alpha}}(n,0) = (r + m\alpha_0)^n$. □

Theorem 5. *The generalized r-*Whitney *numbers of the second kind have the exponential generating function*

$$\varphi_k(t;\bar{\alpha}) = \sum_{n=0}^{\infty} W_{m,r;\bar{\alpha}}(n,k) \frac{t^n}{n!} = \sum_{i=0}^{k} \frac{e^{(r+m\alpha_i)t}}{\prod_{j=0, j\neq i}^{k} (m\alpha_i - m\alpha_j)}. \tag{23}$$

Proof. The exponential generating function of $W_{m,r;\bar{\alpha}}(n,k)$ is defined by

$$\varphi_k(t;\bar{\alpha}) = \sum_{n=k}^{\infty} W_{m,r;\bar{\alpha}}(n,k) \frac{t^n}{n!} \tag{24}$$

where $W_{m,r;\bar{\alpha}}(n,k) = 0$ for $n < k$. If $k = 0$ we have

$$\varphi_0(t;\bar{\alpha}) = \sum_{n=0}^{\infty} W_{m,r;\bar{\alpha}}(n,0) \frac{t^n}{n!} = \sum_{n=0}^{\infty} (r + m\alpha_0)^n \frac{t^n}{n!} = e^{(r+m\alpha_0)t}.$$

Differentiating both sides of Equation (24) with respect to t, we get

$$\dot{\varphi}_k(t;\bar{\alpha}) = \sum_{n=k}^{\infty} W_{m,r;\bar{\alpha}}(n,k) \frac{t^{n-1}}{(n-1)!} \tag{25}$$

and from Equation (22) we have

$$\dot{\varphi}_k(t;\bar{\alpha}) = \sum_{n=k}^{\infty} W_{m,r;\bar{\alpha}}(n-1,k-1) \frac{t^{n-1}}{(n-1)!} + (r + m\alpha_k) \sum_{n=k}^{\infty} W_{m,r;\bar{\alpha}}(n-1,k) \frac{t^{n-1}}{(n-1)!}$$
$$= \varphi_{k-1}(t;\bar{\alpha}) + (r + m\alpha_k) \varphi_k(t;\bar{\alpha}).$$

The solution of this difference-differential equation is

$$I_k(t) \varphi_k(t;\bar{\alpha}) = \int I_k(t) \varphi_{k-1}(t;\bar{\alpha}) dt, \tag{26}$$

where

$$I_k(t) = e^{\int -(r+m\alpha_k)dt} = e^{-(r+m\alpha_k)t}. \tag{27}$$

Setting $k = 1$ in Equation (26) and Equation (27), we get

$$\begin{aligned} e^{-(r+m\alpha_1)t} \varphi_1(t;\bar{\alpha}) &= \int e^{-(r+m\alpha_1)t} \varphi_0(t;\bar{\alpha}) dt \\ &= \int e^{-(r+m\alpha_1)t} e^{(r+m\alpha_0)t} dt \\ &= \int e^{(m\alpha_0 - m\alpha_1)t} dt \\ &= \frac{e^{(m\alpha_0 - m\alpha_1)t}}{m\alpha_0 - m\alpha_1} + c, \end{aligned} \tag{28}$$

if $t=0$ then $c=\dfrac{-1}{m\alpha_0-m\alpha_1}$, substituting in Equation (28), we get

$$\varphi_1(t;\bar{\boldsymbol{\alpha}})=\frac{\mathrm{e}^{(r+m\alpha_0)t}}{m(\alpha_0-\alpha_1)}+\frac{\mathrm{e}^{(r+m\alpha_1)t}}{m(\alpha_1-\alpha_0)}. \tag{29}$$

Similarly at $k=2,3,$ we get

$$\begin{aligned}\varphi_2(t;\bar{\boldsymbol{\alpha}})&=\frac{\mathrm{e}^{(r+m\alpha_0)t}}{m^2(\alpha_0-\alpha_1)(\alpha_0-\alpha_2)}+\frac{\mathrm{e}^{(r+m\alpha_1)t}}{m^2(\alpha_1-\alpha_0)(\alpha_1-\alpha_2)}\\&\quad+\frac{\mathrm{e}^{(r+m\alpha_2)t}}{m^2(\alpha_2-\alpha_0)(\alpha_2-\alpha_1)},\end{aligned} \tag{30}$$

and

$$\begin{aligned}\varphi_3(t;\bar{\boldsymbol{\alpha}})&=\frac{\mathrm{e}^{(r+m\alpha_0)t}}{m^3\prod_{i=1}^{3}(\alpha_0-\alpha_i)}+\frac{\mathrm{e}^{(r+m\alpha_1)t}}{m^3\prod_{i=0,\,i\neq1}^{3}(\alpha_1-\alpha_i)}\\&\quad+\frac{\mathrm{e}^{(r+m\alpha_2)t}}{m^3\prod_{i=0,\,i\neq2}^{3}(\alpha_2-\alpha_i)}+\frac{\mathrm{e}^{(r+m\alpha_3)t}}{m^3\prod_{i=0}^{2}(\alpha_3-\alpha_i)},\end{aligned} \tag{31}$$

by iteration we get Equation (23). □

Theorem 6. *The generalized r-*Whitney *numbers of the second kind have the explicit formula*

$$W_{m,r;\bar{\boldsymbol{\alpha}}}(n,k)=\sum_{i=0}^{k}\frac{1}{m^k(\alpha_i)_k}(r+m\alpha_i)^n. \tag{32}$$

Proof. From Equation (23), we get

$$\begin{aligned}\sum_{n=0}^{\infty}W_{m,r;\bar{\boldsymbol{\alpha}}}(n,k)\frac{t^n}{n!}&=\sum_{i=0}^{k}\frac{1}{m^k(\alpha_i)_k}\sum_{n=0}^{\infty}(r+m\alpha_i)^n\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{k}\frac{1}{m^k(\alpha_i)_k}(r+m\alpha_i)^n\right)\frac{t^n}{n!}.\end{aligned}$$

Equating the coefficient of t^n on both sides, we get Equation (32).

Special cases:

1. Setting $\alpha_i=0$, for $i=0,1,\cdots,n-1$, hence Equation (21) is reduced to

$$\sum_{k=0}^{n}\binom{n}{k}(mx)^k r^{n-k}=\sum_{k=0}^{n}W_{m,r;\mathbf{0}}(n,k)m^k x^k. \tag{33}$$

Equating the coefficients of x^k on both sides, we get

$$\binom{n}{k}r^{n-k}=W_{m,r;\mathbf{0}}(n,k), \tag{34}$$

where $W_{m,r;\mathbf{0}}(n,k)$ denotes the generalized Pascal numbers, for more details see [13], [14].

2. Setting $\alpha_i=\alpha$, for $i=0,1,\cdots,n-1$, hence Equation (21) is reduced to

$$(mx+r)^n=\sum_{k=0}^{n}W_{m,r;\boldsymbol{\alpha}}(n,k)m^k(x-\alpha)^k, \tag{35}$$

hence we have

$$\sum_{i=0}^{n}\binom{n}{i}(mx)^{i}r^{n-i}=\sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i}W_{m,r;\boldsymbol{\alpha}}(n,k)(\alpha)^{k-i}m^{k}\right)x^{i}. \tag{36}$$

Equating the coefficients of x^i on both sides, we get

$$\binom{n}{i}r^{n-i}m^{i}=\sum_{k=i}^{n}\binom{k}{i}W_{m,r;\boldsymbol{\alpha}}(n,k)(\alpha)^{k-i}m^{k}. \tag{37}$$

3. Setting $\alpha_i=i$, for $i=0,1,\cdots,n-1$, hence Equation (21) is reduced to

$$(mx+r)^{n}=\sum_{k=0}^{n}W_{m,r;\boldsymbol{i}}(n,k)m^{k}(x)_{k}, \tag{38}$$

where $W_{m,r;\boldsymbol{i}}(n,k)=W_{m,r}(n,k)$ are the r-Whitney numbers of the second kind.

Remark 7 *Setting* $\alpha_i=i$, $i=0,1,\cdots,n-1$ *in Equation* (23) *and using the identity* $\prod_{j=0,\,j\neq i}^{k}(i-j)=(-1)^{k-i}(k-i)!\,j!$, *given by Gould* [15], *we obtain the exponential generating function of r*-Whitney *numbers of the second kind, see* [1], [3].

4. Setting $\alpha_i=i$, for $i=0,1,\cdots,n-1$, and $r=-a$ hence Equation (21) is reduced to the noncentral Whitney numbers of the second kind see, [9].

5. Setting $\alpha_i=-i\alpha$, for $i=0,1,\cdots,n-1$, hence Equation (21) is reduced to

$$x^{n}=\sum_{k=0}^{n}W_{1,0;-\boldsymbol{i\alpha}}(n,k)(x;-\alpha)_{k}, \tag{39}$$

where $-\boldsymbol{i\alpha}=(0,-\alpha,\cdots,-(n-1)\alpha)$ and $W_{1,0;-\boldsymbol{i\alpha}}(n,k)=\left\{{n \atop k}\right\}^{(\alpha)}$ are the translated Whitney numbers of the second kind defined by Belbachir and Bousbaa [10].

6. Setting $\alpha_i=\binom{i}{p}$, for $i=p-1,p,\cdots,n+p-2$, hence Equation (21) is reduced to

$$(mx+r)^{n}=\sum_{k=0}^{n}W_{m,r;\boldsymbol{p}}(n,k)m^{k}\prod_{i=p-1}^{p+k-1}\left(x-\binom{i}{p}\right). \tag{40}$$

Sun [11] defined the p-Stirling numbers of the second kind as

$$\sum_{k=1}^{n}s_{2}(n,k,p)x^{i}=\prod_{i=p-1}^{p+n-2}\left(x-\binom{n}{i}\right),$$

hence we have

$$\sum_{i=1}^{n}\binom{n}{i}r^{n-i}m^{i}x^{i}=\sum_{i=1}^{n}\left(\sum_{k=i-1}^{n}W_{m,r;p}(n,k)m^{k}s_{2}(k,i,p)\right)x^{i}.$$

Equating the coefficients of x^i on both sides, we get the identity

$$\binom{n}{i}r^{n-i}m^{i}=\sum_{k=i-1}^{n}W_{m,r;\boldsymbol{p}}(n,k)m^{k}s_{2}(k,i,p),$$

where $\boldsymbol{p}=(p-1,p,\cdots,n+p-2)$.

7. Setting $\alpha_i=[\alpha_i]_q$, and $x=[x]_q$ for $i=0,1,\cdots,n-1$, hence Equation (21) is reduced to

$$\left(m[x]_{q}+r\right)^{n}=\sum_{k=0}^{n}W_{m,r;[\bar{\alpha}]_{q}}(n,k)m^{k}\left([x]_{q}-[\alpha]_{q}\right)_{\underline{k}}, \tag{41}$$

El-Desouky and Gomaa [12] defined the generalized q-Stirling numbers of the second kind as

$$[x]_q^i = \sum_{k=0}^{i} q^{\sum_{l=0}^{k-1}\alpha l} S_{q,\bar{\alpha}}(i,k)[x;\bar{\boldsymbol{\alpha}}]_{\underline{k},q}$$
$$= \sum_{k=0}^{i} S_{q,\bar{\alpha}}(i,k)\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_{\underline{k}},$$

therefore we have

$$\sum_{k=0}^{n} W_{m,r;[\bar{\alpha}]_q}(n,k)m^k\left([x]_q-[\alpha]_q\right)_{\underline{k}} = \sum_{i=0}^{n}\binom{n}{i}r^{n-i}m^i[x]_q^i$$
$$= \sum_{i=0}^{n}\binom{n}{i}r^{n-i}m^i\sum_{k=0}^{i}S_{q,\bar{\alpha}}(i,k)\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_{\underline{k}}$$
$$= \sum_{k=0}^{n}\left(\sum_{i=k}^{n}\binom{n}{i}r^{n-i}m^iS_{q,\bar{\alpha}}(i,k)\right)\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_{\underline{k}}.$$

Equating the coefficient of $\left([x]_q;[\bar{\boldsymbol{\alpha}}]_q\right)_{\underline{k}}$ on both sides we get

$$W_{m,r;[\bar{\alpha}]_q}(n,k)m^k = \sum_{i=k}^{n}\binom{n}{i}r^{n-i}m^iS_{q,\bar{\alpha}}(i,k).$$

4. The Generalized Whitney Numbers

When $r=1$, the generalized *r*-Whitney numbers of the first and second kind $w_{m,1;\bar{\alpha}}(n,k)$ and $W_{m,1;\bar{\alpha}}(n,k)$, respectively, are reduced to numbers which we call the generalized Whitney numbers of the first and second kind, which briefly are denoted by $\tilde{w}_{m;\bar{\alpha}}(n,k)$ and $\tilde{W}_{m;\bar{\alpha}}(n,k)$.

4.1. The Generalized Whitney Numbers of the First Kind

Definition 8. *The generalized* Whitney *numbers of the first kind* $\tilde{w}_{m;\bar{\alpha}}(n,k)$ *with parameter* $\bar{\boldsymbol{\alpha}}=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$ are defined by

$$m^n(x;\bar{\boldsymbol{\alpha}})_n = \sum_{k=0}^{n}\tilde{w}_{m;\bar{\alpha}}(n,k)(mx+1)^k, \tag{42}$$

where $\tilde{w}_{m;\bar{\alpha}}(0,0)=1$ and $\tilde{w}_{m;\bar{\alpha}}(n,k)=0$ for $k>n$.

Corollary 1. *The generalized* Whitney *numbers of the first kind* $\tilde{w}_{m;\bar{\alpha}}(n,k)$ *satisfy the recurrence relation*

$$\tilde{w}_{m;\bar{\alpha}}(n,k) = \tilde{w}_{m;\bar{\alpha}}(n-1,k-1)-(1+m\alpha_{n-1})\tilde{w}_{m;\bar{\alpha}}(n-1,k), \tag{43}$$

for $k\ge 1$, and $\tilde{w}_{m;\bar{\alpha}}(n,0)=(-1)^n\prod_{i=1}^{n}(1+m\alpha_{i-1})$.

Proof. The proof follows directly by setting $r=1$ in Equation (6). □

Special cases:

1. Setting $\alpha_i=0$, for $i=0,1,\cdots,n-1$ in Equation (42), we get

$$\sum_{k=i}^{n}\binom{k}{i}\tilde{w}_{m;\bar{0}}(n,k)=\delta_{ni}. \tag{44}$$

2. Setting $\alpha_i=\alpha$, in Equation (42), for $i=0,1,\cdots,n-1$, we get,

$$m^{n-i}\binom{n}{i}(-\alpha)^{n-i} = \sum_{k=i}^{n}\binom{k}{i}\tilde{w}_{m;\alpha}(n,k). \tag{45}$$

3. Setting $\alpha_i=i$, for $i=0,1,\cdots,n-1$, in Equation (42), we get

$$m^n(x)_n=\sum_{k=0}^{n}\tilde{w}_{m;\boldsymbol{i}}(n,k)(mx+1)^k, \tag{46}$$

where $\tilde{w}_{m;\boldsymbol{i}}(n,k)=w_m(n,k)$ are the Whitney numbers of the first kind.

4. Setting $\alpha_i=i^p$, for $i=0,1,\cdots,n-1$, in Equation (42), we get

$$m^{n-i}s_1(n,i,p)=\sum_{k=i}^{n}\binom{k}{i}\tilde{w}_{m;\overline{i^p}}(n,k). \tag{47}$$

5. Setting $\alpha_i=[\alpha_i]_q$ and $x=[x]_q$ for $i=0,1,\cdots,n-1$, in Equation (42), we get

$$m^{n-i}s_{q,\bar{\alpha}}(n,i)=\sum_{k=i}^{n}\tilde{w}_{m;[\bar{\alpha}]_q}(n,k)\binom{k}{i}. \tag{48}$$

4.2. The Generalized Whitney Numbers of the Second Kind

Definition 9. *The generalized* Whitney *numbers of the second kind* $\tilde{W}_{m;\bar{\alpha}}(n,k)$ *with parameter* $\bar{\boldsymbol{\alpha}}=(\alpha_0,\alpha_1,\cdots,\alpha_{n-1})$ are defined by

$$(mx+1)^n=\sum_{k=0}^{n}\tilde{W}_{m;\bar{\alpha}}(n,k)m^k(x;\bar{\boldsymbol{\alpha}})_k, \tag{49}$$

where $\tilde{W}_{m;\bar{\alpha}}(0,0)=1$ and $\tilde{W}_{m;\bar{\alpha}}(n,k)=0$ for $k>n$.

Corollary 2. *The generalized* Whitney *numbers of the second kind* $\tilde{W}_{m;\bar{\alpha}}(n,k)$ *satisfy the recurrence relation*

$$\tilde{W}_{m;\bar{\alpha}}(n,k)=\tilde{W}_{m;\bar{\alpha}}(n-1,k-1)+(1+m\alpha_k)\tilde{W}_{m;\bar{\alpha}}(n-1,k), \tag{50}$$

for $k\ge 1$, and $\tilde{W}_{m;\bar{\alpha}}(n,0)=(1+m\alpha_0)^n$.

Proof. The proof follows directly by setting $r=1$ in Equation (22). □

Corollary 3. *The generalized* Whitney *numbers of the second kind have the exponential generating function*

$$\varphi_k(t;\bar{\boldsymbol{\alpha}})=\sum_{i=0}^{k}\frac{e^{(1+m\alpha_i)t}}{\prod_{j=0,j\ne i}^{k}(m\alpha_i-m\alpha_j)}. \tag{51}$$

Proof. The proof follows directly by setting $r=1$ in Equation (23). □

Corollary 4. *The generalized* Whitney *numbers of the second kind have the explicit formula*

$$\tilde{W}_{m;\bar{\alpha}}(n,k)=\sum_{i=0}^{k}\frac{1}{m^k(\alpha_i)_k}(1+m\alpha_i)^n. \tag{52}$$

Proof. The proof follows directly by setting $r=1$ in Equation (32). □

Special cases:

1. Setting $\alpha_i=0$, for $i=0,1,\cdots,n-1$, in Equation (49), then we get

$$\binom{n}{k}=\tilde{W}_{m;\bar{\mathbf{0}}}(n,k), \tag{53}$$

where $\tilde{W}_{m;\bar{\mathbf{0}}}(n,k)$ are the Pascal numbers.

2. Setting $\alpha_i=\alpha$, for $i=0,1,\cdots,n-1$, in Equation (49), then we get

$$\binom{n}{i}m^i=\sum_{k=i}^{n}\binom{k}{i}\tilde{W}_{m;\alpha}(n,k)m^k. \tag{54}$$

3. Setting $\alpha_i = i$, for $i = 0,1,\cdots,n-1$, in Equation (49), then we get

$$(mx+1)^n = \sum_{k=0}^{n} \tilde{W}_{m;\boldsymbol{i}}(n,k)m^k (x)_k, \tag{55}$$

where $\tilde{W}_{m;\boldsymbol{i}}(n,k) = W_m(n,k)$ are the Whitney numbers of the second kind.

Remark 10. *Setting* $\alpha_i = i$ *and* $r = 1$ *in Equation* (23) *we obtain the exponential generating function of* Whitney *numbers of the second kind, see* [4].

4. Setting $\alpha_i = \binom{i}{p}$, for $i = p-1, p, \cdots, n+p-2$, in Equation (49), we get

$$\binom{n}{i} m^i = \sum_{k=i-1}^{n} \tilde{W}_{m;\boldsymbol{p}}(n,k) m^k s_2(k,i,p).$$

5. Setting $\alpha_i = [\alpha_i]_q$ and $x = [x]_q$ for $i = 0,1,\cdots,n-1$, in Equation (49), we get

$$\tilde{W}_{m;[\bar{\alpha}]_q}(n,k)m^k = \sum_{i=k}^{n} \binom{n}{i} m^i S_{q,\bar{\alpha}}(i,k). \tag{56}$$

5. Relations between Whitney Numbers and Some Types of Numbers

This section is devoted to drive many important relations between the generalized *r*-Whitney numbers and different types of Stirling numbers of the first and second kind and the generalized harmonic numbers.

1. Comtet [7], [16] defined the generalized Stirling numbers of the first and second kind, respectively by,

$$(x;\bar{\alpha})_n = \sum_{i=0}^{n} s_{\bar{\alpha}}(n,i)x^i, \tag{57}$$

$$x^k = \sum_{i=0}^{k} S_{\bar{\alpha}}(k,i)(x;\bar{\alpha})_i, \tag{58}$$

substituting Equation (57) in Equation (5), we obtain

$$m^n \sum_{i=0}^{n} s_{\bar{\alpha}}(n,i)x^i = \sum_{k=0}^{n} w_{m,r;\bar{\alpha}}(n,k)(mx+r)^k$$
$$= \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{k}{i} w_{m,r;\bar{\alpha}}(n,k)r^{k-i}\right)(mx)^i.$$

Equating the coefficients of x^i on both sides, we have

$$s_{\bar{\alpha}}(n,i) = \frac{1}{m^{n-i}} \sum_{k=i}^{n} \binom{k}{i} r^{k-i} w_{m,r;\bar{\alpha}}(n,k). \tag{59}$$

This equation gives the generalized Stirling numbers of the first kind in terms of the generalized *r*-Whitney numbers of the first kind. Moreover, setting $r = 1$, we get

$$s_{\bar{\alpha}}(n,i) = \frac{1}{m^{n-i}} \sum_{k=i}^{n} \binom{k}{i} \tilde{w}_{m;\bar{\alpha}}(n,k). \tag{60}$$

2. From Equation (21) and Equation (58), we have

$$\sum_{i=0}^{n} W_{m,r;\bar{\alpha}}(n,i)m^i (x;\bar{\alpha})_i = \sum_{k=0}^{n}\binom{n}{k} m^k x^k r^{n-k} = \sum_{i=0}^{n}\left(\sum_{k=i}^{n}\binom{n}{k} m^k r^{n-k} S_{\bar{\alpha}}(k,i)\right)(x;\bar{\alpha})_i.$$

Equating the coefficients of $(x;\bar{\boldsymbol{\alpha}})_i$ on both sides, we have

$$W_{m,r;\bar{\alpha}}(n,i)m^i = \sum_{k=i}^{n}\binom{n}{k}m^k r^{n-k} S_{\bar{\alpha}}(k,i), \tag{61}$$

which gives the generalized r-Whitney numbers of the second kind in terms of the generalized Stirling numbers of the second kind. Moreover setting $r=1$, we get

$$\tilde{W}_{m;\bar{\alpha}}(n,i)m^i = \sum_{k=i}^{n}\binom{n}{k}m^k S_{\bar{\alpha}}(k,i). \tag{62}$$

3. El-Desouky [17] defined the multiparameter noncentral Stirling numbers of the first and second kind, respectively by,

$$(x)_n = \sum_{k=0}^{n} s(n,k;\bar{\boldsymbol{\alpha}})(x;\bar{\boldsymbol{\alpha}})_k, \tag{63}$$

$$(x;\bar{\boldsymbol{\alpha}})_n = \sum_{k=0}^{n} S(n,k;\bar{\boldsymbol{\alpha}})(x)_k, \tag{64}$$

using Equation (21) and Equation (2), we have

$$\sum_{k=0}^{n} W_{m,r}(n,k)m^k(x)_k = \sum_{i=0}^{n} W_{m,r;\bar{\alpha}}(n,i)m^i(x;\bar{\boldsymbol{\alpha}})_i, \tag{65}$$

from Equation (63) we get

$$\begin{aligned}\sum_{i=0}^{n} m^i W_{m,r;\bar{\alpha}}(n,i)(x;\bar{\boldsymbol{\alpha}})_i &= \sum_{k=0}^{n} W_{m,r}(n,k)m^k \sum_{i=0}^{k} s(k,i;\bar{\boldsymbol{\alpha}})(x;\bar{\boldsymbol{\alpha}})_i \\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n} m^k W_{m,r}(n,k)s(k,i;\bar{\boldsymbol{\alpha}})\right)(x;\bar{\boldsymbol{\alpha}})_i.\end{aligned}$$

Equating the coefficients of $(x;\bar{\boldsymbol{\alpha}})_i$ on both sides, we have

$$W_{m,r;\bar{\alpha}}(n,i) = \sum_{k=i}^{n} m^{k-i} W_{m,r}(n,k)s(k,i;\bar{\boldsymbol{\alpha}}). \tag{66}$$

This equation gives the generalized r-Whitney numbers of the second kind in terms of r-Whitney numbers of the second kind and the multiparameter noncentral Stirling numbers of the first kind. Moreover setting $r=1$, we get

$$\sum_{k=i}^{n} m^{k-i} W_m(n,k)s(k,i;\bar{\boldsymbol{\alpha}}) = \tilde{W}_{m;\bar{\alpha}}(n,i). \tag{67}$$

4. From Equation (64) and Equation (5), we have

$$\begin{aligned}m^n \sum_{i=0}^{n} S(n,i;\bar{\boldsymbol{\alpha}})(x)_i &= \sum_{k=0}^{n} w_{m,r;\bar{\alpha}}(n,k)(mx+r)^k \\ &= \sum_{i=0}^{n}\left(\sum_{k=i}^{n} w_{m,r;\bar{\alpha}}(n,k)W_{m,r}(k,i)\right)m^i(x)_i.\end{aligned}$$

Equating the coefficients of $(x)_i$ on both sides, we get

$$S(n,i;\bar{\boldsymbol{\alpha}}) = \frac{1}{m^{n-i}}\sum_{k=i}^{n} w_{m,r;\bar{\alpha}}(n,k)W_{m,r}(k,i), \tag{68}$$

which gives the multiparameter noncentral Stirling numbers of the second kind in terms of the generalized r-Whitney numbers of the first kind and r-Whitney numbers of the second kind. Also, setting $r=1,$ we get

$$S(n,i;\bar{\boldsymbol{\alpha}})=\frac{1}{m^{n-i}}\sum_{k=i}^{n}\tilde{w}_{m;\bar{\alpha}}(n,k)W_m(k,i). \tag{69}$$

5. Similarly, from Equation (65) and Equation (64), we get

$$W_{m,r}(n,k)=\sum_{i=k}^{n}W_{m,r;\bar{\alpha}}(n,i)S(i,k;\bar{\boldsymbol{\alpha}})m^{i-k}. \tag{70}$$

Equation (70) gives r-Whitney numbers of the second kind in terms of the multiparameter noncentral Stirling numbers and the generalized r-Whitney numbers of the second kind. Setting $r=1,$ we have

$$W_m(n,k)=\sum_{i=k}^{n}\tilde{W}_{m;\bar{\alpha}}(n,i)S(i,k;\bar{\boldsymbol{\alpha}})m^{i-k}. \tag{71}$$

6. Cakić [18] defined the generalized harmonic numbers as

$$H_n(k;\bar{\boldsymbol{\alpha}})=\sum_{i=0}^{n-1}\frac{1}{(\alpha_i)^k}.$$

From Eq (5), we have

$$\begin{aligned} m^n(x;\bar{\boldsymbol{\alpha}})_n &=\sum_{k=0}^{n}w_{m,r;\bar{\alpha}}(n,k)(mx+r)^k \\ &=\sum_{j=0}^{\infty}\left(\sum_{k=j}^{n}\binom{k}{i}r^{k-j}w_{m,r;\bar{\alpha}}(n,k)\right)(mx)^j. \end{aligned} \tag{72}$$

Also,

$$\begin{aligned}(x;\bar{\boldsymbol{\alpha}})_n &=\prod_{i=0}^{n}(x-\alpha_i)=\prod_{i=0}^{n}(-\alpha_i)\left(1-\frac{x}{\alpha_i}\right)=\prod_{i=0}^{n}(-\alpha_i).\exp\left(\sum_{i=0}^{n-1}\log\left(1-\frac{x}{\alpha_i}\right)\right)\\ &=\prod_{i=0}^{n}(-\alpha_i).\exp\left(-\sum_{k=1}^{\infty}\frac{x^k}{k}\sum_{i=0}^{n-1}\left(\frac{1}{\alpha_i}\right)^k\right)\\ &=\prod_{i=0}^{n}(-\alpha_i).\exp\left(-\sum_{k=1}^{\infty}\frac{x^k}{k}H_n(k;\bar{\boldsymbol{\alpha}})\right)\\ &=\prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\left(\sum_{k=1}^{\infty}\frac{x^k}{k}H_n(k;\bar{\boldsymbol{\alpha}})\right)^\ell,\end{aligned}$$

using Cauchy rule product, this lead to

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\frac{x^k}{k}H_{=}n(k;\bar{\boldsymbol{\alpha}})\right)^\ell &=\prod_{j=1}^{\ell}\left(\sum_{k_j=1}^{\infty}\frac{x^{k_j}}{k_j}H_n(k_j;\bar{\boldsymbol{\alpha}})\right)\\ &=\sum_{j=\ell}^{\infty}\left(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\bar{\boldsymbol{\alpha}})\right)x^j,\end{aligned}$$

therefore, we get

$$\begin{aligned}(x;\bar{\boldsymbol{\alpha}})_n &=\prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\sum_{j=\ell}^{\infty}\left(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\bar{\boldsymbol{\alpha}})\right)x^j\\ &=\prod_{i=0}^{n}(-\alpha_i)\sum_{j=0}^{\infty}\left(\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\left(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\bar{\boldsymbol{\alpha}})\right)\right)x^j.\end{aligned} \tag{73}$$

From Equation (72) and Equation (73) we have the following identity

$$\sum_{k=j}^{n}\binom{k}{i}r^{k-j}w_{m,r;\bar{\alpha}}(n,k) = m^{n-j}\prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\left(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\bar{\boldsymbol{\alpha}})\right) \tag{74}$$

From Equation (59) and Equation (74) we have

$$s_{\bar{\alpha}}(n,j) = \prod_{i=0}^{n}(-\alpha_i)\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}}{\ell!}\left(\sum_{k_1+k_2+\cdots+k_\ell=j}\frac{1}{k_1k_2\cdots k_\ell}\prod_{\iota=1}^{\ell}H_n(k_\iota;\bar{\boldsymbol{\alpha}})\right), \tag{75}$$

this equation gives the generalized Stirling numbers of the first kind in terms of the generalized Harmonic numbers.

6. Matrix Representation

In this section we drive a matrix representation for some given relations.

1. Equation (66) can be represented in matrix form as

$$\hat{\boldsymbol{W}}_{m,r}\,\boldsymbol{s}(\bar{\boldsymbol{\alpha}}) = \hat{\boldsymbol{W}}_{m,r;\bar{\alpha}}, \tag{76}$$

where $\hat{W}_{m,r}(n,k) = m^k W_{m,r}(n,k)$, $\hat{W}_{m,r;\bar{\alpha}}(n,i) = m^i W_{m,r;\bar{\alpha}}(n,i)$ and $\boldsymbol{W}_{m,r}, \boldsymbol{s}(\bar{\boldsymbol{\alpha}})$ and $\boldsymbol{W}_{m,r;\bar{\alpha}}$ are $n\times n$ lower triangle matrices whose entries are, respectively, the r-Whitney numbers of the second kind, the multiparameter noncentral Stirling numbers of the first kind and the generalized r-Whitney numbers of the second kind.

For example if $0\le n,k,i\le 3$, and using matrix representation given in [19], hence Equation (76) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ r & m & 0 & 0 \\ r^2 & m(2r+m) & m^2 & 0 \\ r^3 & m(3r^2+3mr+m^2) & m^2(3r+3m) & m^3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha_0 & 1 & 0 & 0 \\ \alpha_0(\alpha_0-1) & \alpha_0+\alpha_1-1 & 1 & 0 \\ s(3,0;\bar{\boldsymbol{\alpha}}) & s(3,1;\bar{\boldsymbol{\alpha}}) & s(3,2;\bar{\boldsymbol{\alpha}}) & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ (r+m\alpha_0) & m & 0 & 0 \\ (r+m\alpha_0)^2 & (2r+m\alpha_0+m\alpha_1)m & m^2 & 0 \\ (r+m\alpha_0)^3 & \hat{W}_{m,r;\bar{\alpha}}(3,1) & \hat{W}_{m,r;\bar{\alpha}}(3,2) & m^3 \end{pmatrix}$$

where

$$s(3,0;\bar{\boldsymbol{\alpha}}) = \alpha_0(\alpha_0-1)(\alpha_0-2),$$

$$s(3,1;\bar{\boldsymbol{\alpha}}) = \alpha_0(\alpha_0-1)+(\alpha_1-2)(\alpha_0+\alpha_1-1),$$

$$s(3,2;\bar{\boldsymbol{\alpha}}) = \alpha_0+\alpha_1+\alpha_2-3,$$

$$\hat{W}_{m,r;\bar{\alpha}}(3,1) = \left((r+m\alpha_0)^2+(r+m\alpha_1)(2r+m\alpha_0+m\alpha_1)\right)m,$$

$$\hat{W}_{m,r;\bar{\alpha}}(3,2) = (3r+m\alpha_0+m\alpha_1+m\alpha_2)m^2.$$

2. Equation (68) can be represented in a matrix form as

$$\boldsymbol{w}_{m,r;\bar{\boldsymbol{\alpha}}}\hat{\boldsymbol{W}}_{m,r}=\hat{\boldsymbol{S}}(\bar{\boldsymbol{\alpha}}), \tag{77}$$

where $\hat{S}(n,i;\bar{\boldsymbol{\alpha}})=m^nS(n,i;\bar{\boldsymbol{\alpha}})$, and $\boldsymbol{w}_{m,r;\bar{\boldsymbol{\alpha}}}$ and $\boldsymbol{S}(\bar{\boldsymbol{\alpha}})$ are $n\times n$ lower triangle matrices whose entries are, respectively, the generalized r-Whitney numbers of the first kind and the multiparameter noncentral Stirling numbers of the second kind.

For example if $0\le n,k,i\le 3$, hence Equation (77) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ -r-m\alpha_0 & 1 & 0 & 0\\ (r+m\alpha_0)(r+m\alpha_1) & -2r-m\alpha_0-m\alpha_1 & 1 & 0\\ w_{m,r;\bar{\boldsymbol{\alpha}}}(3,0) & w_{m,r;\bar{\boldsymbol{\alpha}}}(3,1) & w_{m,r;\bar{\boldsymbol{\alpha}}}(3,2) & 1\end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0\\ r & m & 0 & 0\\ r^2 & m(2r+m) & m^2 & 0\\ r^3 & m(3r^2+3mr+m^2) & m^2(3r+3m) & m^3\end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 & 0 & 0\\ -m\alpha_0 & m & 0 & 0\\ m^2\alpha_0\alpha_1 & m^2(-\alpha_0-\alpha_1+1) & m^2 & 0\\ m^3\alpha_0\alpha_1\alpha_2 & \hat{S}(3,1;\bar{\boldsymbol{\alpha}}) & \hat{S}(3,2;\bar{\boldsymbol{\alpha}}) & m^3\end{pmatrix}$$

where

$$w_{m,r;\bar{\boldsymbol{\alpha}}}(3,0)=-(r+m\alpha_0)(r+m\alpha_1)(r+m\alpha_2),$$
$$w_{m,r;\bar{\boldsymbol{\alpha}}}(3,1)=(r+m\alpha_0)(r+m\alpha_1)+(2r+m\alpha_0+m\alpha_1)(r+m\alpha_2),$$
$$w_{m,r;\bar{\boldsymbol{\alpha}}}(3,2)=-3r-m\alpha_0-m\alpha_1-m\alpha_2,$$
$$\hat{S}(3,1;\bar{\boldsymbol{\alpha}})=m^3(\alpha_0\alpha_1+\alpha_0\alpha_2+\alpha_1\alpha_2-\alpha_0-\alpha_1-\alpha_2+1),$$
$$\hat{S}(3,2;\bar{\boldsymbol{\alpha}})=m^3(-\alpha_0-\alpha_1-\alpha_2+3).$$

3. Equation (70) can be represented in a matrix form as

$$\hat{\boldsymbol{W}}_{m,r;\bar{\boldsymbol{\alpha}}}\boldsymbol{S}(\bar{\boldsymbol{\alpha}})=\hat{\boldsymbol{W}}_{m,r}, \tag{78}$$

For example if $0\le n,k,i\le 3$, hence Equation (77) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0\\ (r+m\alpha_0) & m & 0 & 0\\ (r+m\alpha_0)^2 & (2r+m\alpha_0+m\alpha_1)m & m^2 & 0\\ (r+m\alpha_0)^3 & \hat{W}_{m,r;\bar{\boldsymbol{\alpha}}}(3,1) & \hat{W}_{m,r;\bar{\boldsymbol{\alpha}}}(3,2) & m^3\end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 0 & 0\\ -\alpha_0 & 1 & 0 & 0\\ \alpha_0\alpha_1 & -\alpha_0-\alpha_1+1 & 1 & 0\\ \alpha_0\alpha_1\alpha_2 & S(3,1;\bar{\boldsymbol{\alpha}}) & S(3,2;\bar{\boldsymbol{\alpha}}) & 1\end{pmatrix}$$
$$=\begin{pmatrix} 1 & 0 & 0 & 0\\ r & m & 0 & 0\\ r^2 & m(2r+m) & m^2 & 0\\ r^3 & m(3r^2+3mr+m^2) & m^2(3r+3m) & m^3\end{pmatrix}$$

where

$$\hat{W}_{m,r;\bar{\alpha}}(3,1)=\left((r+m\alpha_0)^2+(r+m\alpha_1)(2r+m\alpha_0+m\alpha_1)\right)m,$$

$$\hat{W}_{m,r;\bar{\alpha}}(3,2)=(3r+m\alpha_0+m\alpha_1+m\alpha_2)m^2,$$

$$S(3,1;\bar{\alpha})=\alpha_0\alpha_1+\alpha_0\alpha_2+\alpha_1\alpha_2-\alpha_0-\alpha_1-\alpha_2+1,$$

$$S(3,2;\bar{\alpha})=-\alpha_0-\alpha_1-\alpha_2+3.$$

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