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Global Attractors for a Class of Generalized Nonlinear Kirchhoff-Sine-Gordon Equation

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Abstract

In this paper, we consider a class of generalized nonlinear Kirchhoff-Sine-Gordon equation $u_{tt}-\beta\Delta u_t+\alpha u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+g(\sin u)=f(x)$. By a priori estimation, we first prove the existence and uniqueness of solutions to the initial boundary value conditions, and then we study the global attractors of the equation.

Keywords

Kirchhoff-Sine-Gordon Equation, The Existence and Uniqueness of Solutions, Priori Estimates, Global Attractors

1. Introduction

In 1883, Kirchhoff [1] proposed the following model in the study of elastic string free vibration: $u_{tt}-\alpha\Delta u-M\left(\|\nabla u\|^2\right)\Delta u=f(x,u)$, where α is associated with the initial tension, M is related to the material properties of the rope, and $u(x,t)$ indicates the vertical displacement at the x point on the t. The equation is more accurate than the classical wave equation to describe the motion of an elastic rod.

Masamro [2] proposed the Kirchhoff equation with dissipation and damping term:

$$\begin{cases} u_{tt}-M\left(\|\nabla u\|^2\right)\Delta u+\delta|u|^p u+\gamma u_t=f(x) & x\in\Omega, t>0 \\ u(x,t)=0 & x\in\partial\Omega, t\geq 0 \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x) & x\in\Omega \end{cases}$$

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where Ω is a bounded domain of $R^n(n\geq 1)$ with a smooth boundary $\partial\Omega$; he uses the Galerkin method to prove the existence of the solution of the equation at the initial boundary conditions.

Sine-Gordon equation is a very useful model in physics. In 1962, Josephson [3] fist applied the Sine-Gordon equation to superconductors, where the equation: $u_{tt}-u_{xx}+\sin u=0$, u_{tt} is the two-order partial derivative of *u* with respect to the variable *t*; u_{xx} is the two-order partial derivative of the *u* about the independent variable *x*. Subsequently, Zhu [4] considered the following problem: $u_{tt}-\alpha u_t-u_{xx}+\lambda g(\sin u)=f(x,t)$ (where Ω is a bounded domain of R^3) and he proved the existence of the global solution of the equation. For more research on the global solutions and global attractors of Kirchhoff and sine-Gordon equations, we refer the reader to [5]-[11].

Based on Kirchhoff and Sine-Gordon model, we study the following initial boundary value problem:

$$\begin{cases} u_{tt}-\beta\Delta u_t+\alpha u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+g(\sin u)=f(x) & \\ u(x,t)=0 & x\in\partial\Omega, t\geq 0 \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x) & x\in\Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $R^n(n\geq 1)$ with a smooth boundary $\partial\Omega$; α is the dissipation coefficient; β is a positive constant; and $f(x)$ is the external interference. The assumptions on nonlinear terms $g(\sin u)$ and $\phi\left(\|\nabla u\|^2\right)$ will be specified later.

The rest of this paper is organized as follows. In Section 2, we first obtain the basic assumption. In Section 3, we obtain a priori estimate. In Section 4, we prove the existence of the global attractors.

2. Basic Assumption

For brevity, we define the Sobolev space as follows:

$$H=L^2(\Omega),V_1=H_0^1(\Omega),V_2=H^2(\Omega)\cap H_0^1(\Omega),$$

$$E_0=H_0^1(\Omega)\times L^2(\Omega)=V_1\times H, E_1=\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)=V_2\times V_1.$$

In addition, we define $(\bullet,\bullet)$ and $\|\bullet\|$ are the inner product and norm of *H*.

Nonlinear function $g(s)$ satisfying condition (*G*):

(1) $g(s)\in C^2(R)$;

(2) $|g(s)|\leq c\left(1+|s|^p\right)$;

(3) $\left|\dfrac{\mathrm{d}g(s)}{\mathrm{d}s}\right|\leq c\left(1+|s|^{p-1}\right)$, where $c>0, 1\leq p\leq \dfrac{2n}{n-2}, n\geq 3$.

Function $\phi(s)$ satisfies the condition (*F*):

(4) $\phi(s)\in C^1([0,+\infty),R)$;

(5) $\dfrac{m_1+2}{2}<m_0\leq\phi(s)\leq m_1, 0\leq\dfrac{\mathrm{d}\phi(s)}{\mathrm{d}s}\leq c_0$;

(6) $\Phi(s)=\int_0^u\phi(\tau)\mathrm{d}\tau$;

(7) $\phi(s)s\geq c_1\Phi(s)$, where $c_1\geq\dfrac{2(m+1)}{m_0}, m=\begin{cases} m_0, \dfrac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2\geq 0 \\ m_1, \dfrac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2<0. \end{cases}$

3. A Priori Estimates

Lemma 3.1. Assuming the nonlinear function $g(s),\phi(s)$ satisfies the condition (*G*)-(*F*), $(u_0,u_1)\in V_1\times H$,

$f \in H, v = u_t + \varepsilon u, 0 < \varepsilon \le \min\left\{\frac{\alpha}{4}, \frac{m_0}{2\beta}, \frac{2m_0 - m - 2}{3\beta}\right\}$, then the solution (u,v) of the initial boundary value problem (1.1) satisfies $(u,v) \in V_1 \times H$ and

$$\|(u,v)\|_{V_1\times H}^2 = \|\nabla u\|^2 + \|v\|^2 \le \frac{y_1(0)}{k_1}\mathrm{e}^{-\alpha_1 t} + \frac{c_2}{\alpha_1 k_1}\left(1 - \mathrm{e}^{-\alpha_1 t}\right),$$

where $y_1(0) = \|v(0)\|^2 + \Phi\left(\|\nabla u(0)\|^2\right) - \beta\varepsilon\|\nabla u(0)\|^2$. Thus there exists a positive constant $c(R_0)$ and $t_1 = t_1(\Omega) > 0$, such that

$$\|(u,v)\|_{V_1\times H}^2 = \|\nabla u(t)\|^2 + \|v(t)\|^2 \le c(R_0)\,(t > t_1).$$

Proof. Let $v = u_t + \varepsilon u$, the equation $u_{tt} - \beta\Delta u_t + \alpha u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + g(\sin u) = f(x)$ can be transformed into

$$v_t + (\alpha - \varepsilon)v + \varepsilon(\varepsilon - \alpha)u + \beta\varepsilon\Delta u - \beta\Delta v - \phi\left(\|\nabla u\|^2\right)\Delta u + g(\sin u) = f(x). \tag{3.1}$$

Taking the inner product of the equations (3.1) with *v* in *H*, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + (\alpha - \varepsilon)\|v\|^2 + \varepsilon(\varepsilon - \alpha)(u,v) - \frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2\\ &-\beta\varepsilon^2\|\nabla u\|^2 + (-\beta\Delta v, v) + \left(-\phi\left(\|\nabla u\|^2\right)\Delta u, v\right) = (f - g(\sin u), v).\end{aligned} \tag{3.2}$$

By using Holder inequality, Young's inequality and Poincare inequality, we deal with the terms in (3.2) one by as follows

$$(-\beta\Delta v, v) = \beta(\nabla v, \nabla v) = \beta\|\nabla u\|^2 \ge \lambda_1\beta\|u\|^2, \tag{3.3}$$

where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions on Ω.

Since $0 < \varepsilon \le \frac{\alpha}{4}$ and (F) (6), (7), we get

$$\begin{aligned}\varepsilon(\varepsilon - \alpha)(u,v) &\ge \frac{\varepsilon(\varepsilon - \alpha)}{\sqrt{\lambda_1}}\|\nabla u\|\|v\| \ge -\frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\left(\frac{\sqrt{\lambda_1}}{\alpha}\|\nabla u\|^2 + \frac{\alpha}{\sqrt{\lambda_1}}\|v\|^2\right)\\ &\ge -\varepsilon\|\nabla u\|^2 - \frac{\alpha^3}{4\lambda_1}\|v\|^2,\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\left(-\phi\left(\|\nabla u\|^2\right)\Delta u, v\right) &= \frac{\phi\left(\|\nabla u\|^2\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \varepsilon\phi\left(\|\nabla u\|^2\right)\|\nabla u\|^2\\ &\ge \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Phi\left(\|\nabla u\|^2\right) + c_1\varepsilon\Phi\left(\|\nabla u\|^2\right).\end{aligned} \tag{3.5}$$

$$(f - g(\sin u), v) \le \|v\|\left(\|f\| + \|g(\sin u)\|\right) \le \frac{\alpha}{2}\|v\|^2 + \frac{\left(\|f\| + 2c|\Omega|^{\frac{1}{2}}\right)^2}{2\alpha}, \tag{3.6}$$

where

$$\|g(\sin u)\| \le \left(\int_\Omega \left|c\left(1 + |\sin u|^p\right)\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} \le 2c|\Omega|^{\frac{1}{2}}. \tag{3.7}$$

Combined (3.1)-(3.6) type, it follows from that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2\right]+\left(\alpha+2\lambda_1\beta-\frac{\alpha^3}{2\lambda_1}-2\varepsilon\right)\|v\|^2$$
$$+2c_1\varepsilon\Phi\left(\|\nabla u\|^2\right)+2\varepsilon(-\beta\varepsilon-1)\|\nabla u\|^2\le\frac{\left(\|f\|+2c|\Omega|^{\frac{1}{2}}\right)^2}{\alpha}=c_2. \tag{3.8}$$

According to condition (F) (5), this will imply $m_0\|\nabla u\|^2\le\Phi\left(\|\nabla u\|^2\right)\le m_1\|\nabla u\|^2$, then, $\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2>0$, and since $c_1\ge\frac{2(m+1)}{m_0}$,

$$c_1\Phi\left(\|\nabla u\|^2\right)+(-\beta\varepsilon-1)\|\nabla u\|^2\ge\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2+(m_0-1)\|\nabla u\|^2$$
$$\ge\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2, \tag{3.9}$$

that is

$$2\varepsilon\left[c_1\Phi\left(\|\nabla u\|^2\right)+(-\beta\varepsilon-1)\|\nabla u\|^2\right]>\varepsilon\left[\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2\right]. \tag{3.10}$$

With (3.10), (3.8) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2\right]+\left(\alpha+2\lambda_1\beta-\frac{\alpha^3}{2\lambda_1}-2\varepsilon\right)\|v\|^2$$
$$+\varepsilon\left[\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2\right]\le c_2. \tag{3.11}$$

Set $a=\alpha+2\lambda_1\beta-\frac{\alpha^3}{2\lambda_1}-2\varepsilon\ge 0$, and $\alpha_1=\min\{a,\varepsilon\}$, then (3.11) is equivalent to (3.12)

$$\frac{\mathrm{d}}{\mathrm{d}t}y_1(t)+\alpha_1y_1(t)\le c_2, \tag{3.12}$$

where

$$y_1(t)=\|v\|^2+\Phi\left(\|\nabla u\|^2\right)-\beta\varepsilon\|\nabla u\|^2. \tag{3.13}$$

By using Gronwall inequality, we obtain

$$y_1(t)\le y_1(0)\mathrm{e}^{-\alpha_1t}+\frac{c_2}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1t}\right). \tag{3.14}$$

Let $k_1=\min\{1,(m_0-\beta\varepsilon)\}$.

So, we have

$$\|(u,v)\|^2_{V_1\times H}=\|\nabla u\|^2+\|v\|^2\le\frac{y_1(0)}{k_1}\mathrm{e}^{-\alpha_1t}+\frac{c_2}{\alpha_1k_1}\left(1-\mathrm{e}^{-\alpha_1t}\right), \tag{3.15}$$

then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{V_1\times H}\le\frac{c_2}{\alpha_1k_1}. \tag{3.16}$$

Hence, there exists $c(R_0)$ and $t_1=t_1(\Omega)>0$, such that

$$\|(u,v)\|^2_{V_1\times H}=\|\nabla u(t)\|^2+\|v(t)\|^2\le c(R_0)(t>t_1).$$ ■

Lemma 3.2. Assuming the nonlinear function $g(s),\phi(s)$ satisfies the condition (*G*)-(*F*), $(u_0,u_1)\in V_2\times V_1,\ f\in V_1, v=u_t+\varepsilon u, 0<\varepsilon\le\min\left\{\frac{\alpha}{4},\frac{m_0}{2\beta},\frac{2m_0-m-2}{3\beta}\right\}$, then the solution (u,v) of satisfies the initial boundary value problem (1.1) satisfies $(u,v)\in V_2\times V_1$ and

$$\|(u,v)\|^2_{V_2\times V_1}=\|\nabla v\|^2+\|\Delta u\|^2\le\frac{y_2(0)}{k_2}\mathrm{e}^{-\alpha_2 t}+\frac{c_3}{\alpha_2 k_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right),$$

where $y_2(0)=\|\nabla v(0)\|^2+(m-\beta\varepsilon)\|\Delta u(0)\|^2$. Thus there exists a positive constant $c(R_1)$ and $t_2=t_2(\Omega)>0$, such that

$$\|(u,v)\|^2_{V_2\times V_1}=\|\nabla v(t)\|^2+\|\Delta u(t)\|^2\le c(R_1)(t>t_1).$$

Proof. The equations (3.1) in the H and $-\Delta v=-\Delta u_t-\varepsilon\Delta u$ have inner product, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla v\|^2+(\alpha-\varepsilon)\|\nabla v\|^2+\varepsilon(\varepsilon-\alpha)(u,-\Delta v)-\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2-\beta\varepsilon^2\|\Delta u\|^2\\&+(-\beta\Delta v,-\Delta v)+\left(-\phi\left(\|\nabla u\|^2\right)\Delta u,-\Delta v\right)=\left(f-g(\sin u),-\Delta v\right).\end{aligned}\tag{3.17}$$

By using Holder inequality, Young's inequality and Poincare inequality, we get the following results

$$(-\beta\Delta v,-\Delta v)=\beta(\Delta v,\Delta v)=\beta\|\Delta v\|^2\ge\lambda_1\beta\|\nabla v\|^2,\tag{3.18}$$

$$\begin{aligned}\varepsilon(\varepsilon-\alpha)(u,-\Delta v)&\ge\frac{\varepsilon^2-\varepsilon\alpha}{\sqrt{\lambda_1}}\|\Delta u\|\|\nabla v\|\ge-\frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\left(\frac{\sqrt{\lambda_1}}{\alpha}\|\Delta u\|^2+\frac{\alpha}{\sqrt{\lambda_1}}\|\nabla v\|^2\right)\\&\ge-\varepsilon\|\Delta u\|^2-\frac{\alpha^3}{4\lambda_1}\|\nabla v\|^2.\end{aligned}\tag{3.19}$$

According to condition (F) (5), (6), we obtain

$$\begin{aligned}\left(-\phi\left(\|\nabla u\|^2\right)\Delta u,-\Delta v\right)&=\phi\left(\|\nabla u\|^2\right)(\Delta u,\Delta v)=\phi\left(\|\nabla u\|^2\right)\left[(\Delta u,\Delta u_t)+(\Delta u,\varepsilon\Delta u)\right]\\&=\frac{\phi\left(\|\nabla u\|^2\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2+\varepsilon\phi\left(\|\nabla u\|^2\right)\|\Delta u\|^2\ge\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u\|^2+\varepsilon m_0\|\Delta u\|^2.\end{aligned}\tag{3.20}$$

$$\begin{aligned}\left(f-g(\sin u),-\Delta v\right)&\le\|\nabla v\|\left(\|\nabla f\|+\|\nabla g(\sin u)\|\right)\le\frac{\alpha}{2}\|\nabla v\|^2+\frac{\left(\|\nabla f\|+\|\nabla g(\sin u)\|\right)^2}{2\alpha}\\&\le\frac{\alpha}{2}\|\nabla v\|^2+\frac{\left(\|\nabla f\|+2c|\Omega|^{\frac{1}{2}}\right)^2}{2\alpha},\end{aligned}\tag{3.21}$$

where

$$\begin{aligned}\|\nabla g(\sin u)\|&=\|g'(\sin u)\cos u\nabla u\|\le\|g'(\sin u)\|\|\nabla u\|\\&\le R_1\left(\int_\Omega\left|c\left(1+|\sin u|^{p-1}\right)\right|^2\mathrm{d}x\right)^{\frac{1}{2}}\le 2R_1c|\Omega|^{\frac{1}{2}}.\end{aligned}\tag{3.22}$$

By (3.18)-(3.22), (3.17) can be written

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla v\|^2+(m-\beta\varepsilon)\|\Delta u\|^2\right]+\left(\alpha+2\lambda_1\beta-2\varepsilon-\frac{\alpha^3}{2\lambda_1}\right)\|\nabla v\|^2\\&+2\varepsilon\left(m_0-\beta\varepsilon-1\right)\|\Delta u\|^2\le\frac{\left(\|\nabla f\|+2cR_1|\Omega|^{\frac{1}{2}}\right)^2}{\alpha}=c_3.\end{aligned}\tag{3.23}$$

Noticing $0<\varepsilon\le\frac{2m_0-m-2}{3\beta}$, this will imply

$$2\varepsilon\left(m_0-\beta\varepsilon-1\right)\|\Delta u\|^2\ge\varepsilon\left(m-\beta\varepsilon\right)\|\Delta u\|^2. \tag{3.24}$$

Substituting (3.24) into (3.23), we can get the following inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla v\|^2+\left(m-\beta\varepsilon\right)\|\Delta u\|^2\right]+\left(\alpha+2\lambda_1\beta-2\varepsilon-\frac{\alpha^3}{2\lambda_1}\right)\|\nabla v\|^2+\varepsilon\left(m-\beta\varepsilon\right)\|\Delta u\|^2\le c_3. \tag{3.25}$$

Let $b=\alpha+2\lambda_1\beta-2\varepsilon-\frac{\alpha^3}{2\lambda_1}\ge 0$, and $\alpha_2=\min\{b,\varepsilon\}$, then (3.25) type can be changed into

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla v\|^2+\left(m-\beta\varepsilon\right)\|\Delta u\|^2\right]+\alpha_2\left[\|\nabla v\|^2+\left(m-\beta\varepsilon\right)\|\Delta u\|^2\right]\le c_3, \tag{3.26}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}y_2(t)+\alpha_2 y_2(t)\le c_3, \tag{3.27}$$

where $y_2(t)=\|\nabla v\|^2+\left(m-\beta\varepsilon\right)\|\Delta u\|^2$.

By using Gronwall inequality, we obtain

$$y_2(t)\le y_2(0)\mathrm{e}^{-\alpha_2 t}+\frac{c_3}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right), \tag{3.28}$$

taking $k_2=\min\{1,(m-\beta\varepsilon)\}$, we have

$$\|(u,v)\|^2_{V_2\times V_1}=\|\Delta u\|^2+\|\nabla v\|^2\le\frac{y_2(0)}{k_2}\mathrm{e}^{-\alpha_2 t}+\frac{c_3}{\alpha_2 k_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right), \tag{3.29}$$

then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{V_2\times V_1}\le\frac{c_3}{\alpha_2 k_2}. \tag{3.30}$$

Hence, there exists $c(R_1)$ and $t_2=t_2(\Omega)>0$, such that

$$\|(u,v)\|^2_{V_2\times V_1}=\|\Delta u(t)\|^2+\|\nabla v(t)\|^2\le c(R_1)(t>t_2).$$ ■

Theorem 3.1. Assuming the nonlinear function $g(s),\phi(s)$ satisfies the condition (*G*)-(*F*), $(u_0,u_1)\in V_2\times V_1,\ f\in V_1, v=u_t+\varepsilon u$, $0<\varepsilon\le\min\left\{\frac{\alpha}{4},\frac{m_0}{2\beta},\frac{2m_0-m-2}{3\beta}\right\}$, so the initial boundary value problem (1.1) exists a unique smooth solution $(u,v)\in L^\infty([0,+\infty);V_2\times V_1)$.

Proof. By Lemma 3.1-Lemma 3.2 and Glerkin method, we can easily obtain the existence of solutions of equation $(u,v)\in L^\infty([0,+\infty);V_2\times V_1)$, the proof procedure is omitted. Next, we prove the uniqueness of solutions in detail.

Assume u,v are two solutions of equation, we denote $w=u-v$, then, the two equations subtract and obtain

$$w_{tt}-\beta\Delta w_t+\alpha w_t-\phi\left(\|\nabla u\|^2\right)\Delta u+\phi\left(\|\nabla v\|^2\right)\Delta v=-g(\sin u)+g(\sin v). \tag{3.31}$$

We take the inner product of the above equations (3.31) with w_t in *H*, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2+(-\beta\Delta w_t,w_t)+\alpha\|w_t\|^2+\left(-\phi\left(\|\nabla u\|^2\right)\Delta u+\phi\left(\|\nabla v\|^2\right)\Delta v,w_t\right)$$
$$=\left(-g(\sin u)+g(\sin v),w_t\right). \tag{3.32}$$

We deal with the terms in (3.32) one by as follows

$$(-\beta\Delta w_t,w_t)=\beta(\nabla w_t,\nabla w_t)=\beta\|\nabla w_t\|^2\ge\lambda_1\beta\|w_t\|^2, \tag{3.33}$$

and

$$\begin{aligned}&\left(-\phi\left(\|\nabla u\|^2\right)\Delta u+\phi\left(\|\nabla v\|^2\right)\Delta v,w_t\right)\\&=\left(-\phi\left(\|\nabla u\|^2\right)\Delta u+\phi\left(\|\nabla u\|^2\right)\Delta v-\phi\left(\|\nabla u\|^2\right)\Delta v+\phi\left(\|\nabla v\|^2\right)\Delta v,w_t\right)\\&=-\phi\left(\|\nabla u\|^2\right)(\Delta u-\Delta v,w_t)+\left(-\phi\left(\|\nabla u\|^2\right)+\phi\left(\|\nabla v\|^2\right)\right)(\Delta v,w_t)\\&=\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2+\left(-\phi\left(\|\nabla u\|^2\right)+\phi\left(\|\nabla v\|^2\right)\right)(\Delta v,w_t).\end{aligned} \tag{3.34}$$

By (3.32)-(3.34), we can get the following inequality

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_t\|^2+\lambda_1\beta\|w_t\|^2+\alpha\|w_t\|^2+\frac{m}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla w\|^2\\&=\left(\phi\left(\|\nabla u\|^2\right)-\phi\left(\|\nabla v\|^2\right)\right)(\Delta v,w_t)+\left(-g(\sin u)+g(\sin v),w_t\right).\end{aligned} \tag{3.35}$$

Further, by mid-value theorem and Young's inequality, we get

$$\begin{aligned}&\left(\phi\left(\|\nabla u\|^2\right)-\phi\left(\|\nabla v\|^2\right)\right)(\Delta v,w_t)\le|\phi'(\eta)|\left(\|\nabla u\|+\|\nabla v\|\right)\|\nabla w\|\|\Delta v\|\|w_t\|\\&\le c_0\left(\|\nabla u\|+\|\nabla v\|\right)\|\nabla w\|\|\Delta v\|\|w_t\|\le c_4\|\nabla w\|\|w_t\|\le\frac{c_4}{2\lambda_1}\|\nabla w\|^2+\frac{\lambda_1c_4}{2}\|w_t\|^2.\end{aligned} \tag{3.36}$$

Since $\|g(\sin v)-g(\sin u)\|^2=\int_\Omega\left(\frac{g(\sin v)-g(\sin u)}{\sin v-\sin u}\right)^2\cdot(\sin v-\sin u)^2\,\mathrm{d}x=\int_\Omega\left(g'(\gamma)\right)^2(\sin v-\sin u)^2\,\mathrm{d}x$,
might as well set $\gamma=\theta\sin v+(1-\theta)\sin u\ (0\le\theta\le1)$.

$$\begin{aligned}\|g(\sin v)-g(\sin u)\|^2&\le\int_\Omega\left[c\left(1+|\theta\sin v+(1-\theta)\sin u|^{p-1}\right)\right]^2(\sin v-\sin u)^2\,\mathrm{d}x\\&\le c^2\left(1+2^{p-1}\right)^2\int_\Omega(\sin v-\sin u)^2\,\mathrm{d}x\le c_5^2\|u-v\|^2,\end{aligned}$$

where $c_5=c\left(1+2^{p-1}\right)$.

Then, we obtain

$$\left(-g(\sin u)+g(\sin v),w_t\right)\le c_5\|w\|\|w_t\|\le\frac{c_5}{2\lambda_1}\|\nabla w\|^2+\frac{c_5}{2}\|w_t\|^2. \tag{3.37}$$

Substituting (3.36), (3.37) into (3.35), we can get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+m\|\nabla w\|^2\right)\le\frac{c_4+c_5}{\lambda_1}\|\nabla w\|^2+(c_5+\lambda_1c_4)\|w_t\|^2. \tag{3.38}$$

Let $k=\max\left\{(c_5+\lambda_1c_4),\frac{c_4+c_5}{\lambda_1m}\right\}$, then (3.38) can be changed to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|w_t\|^2+m\|\nabla w\|^2\right)\le k\left(\|w_t\|^2+m\|\nabla w\|^2\right). \tag{3.39}$$

By using Gronwall inequality, we obtain

$$\left\|w_t(t)\right\|^2 + m\left\|\nabla w(t)\right\|^2 \le \left\|w_t(0)\right\|^2 + m\left\|\nabla w(0)\right\|^2 \mathrm{e}^{kt} = 0. \tag{3.40}$$

There has

$$\left\|w_t(t)\right\|^2 + m\left\|\nabla w(t)\right\|^2 \le 0. \tag{3.41}$$

That show that $w_t(t) = 0, \nabla w(t) = 0$.

So as to get $w(t) \equiv 0, u = v$, the uniqueness is proved. ■

4. Global Attractor

Theorem 4.1. [12] Set E_1 be a Banach space, and $\{S(t)\}(t \ge 0)$ are the semigroup operator on E_1. $S(t): E_1 \to E_1, S(t+s) = S(t)S(s)(\forall t, s \ge 0), S(0) = I$; here I is a unit operator. Set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is bounded, namely $\forall R > 0, \|u\|_{E_1} \le R$; it exists a constant $c(R)$, so that

$$\|S(t)u\|_{E_1} \le c(R)(t \in [0, +\infty));$$

2) It exists a bounded absorbing set $B_0 \subset E_1$, namely, $\forall B \subset E_1$; it exists a constant t_0, so that

$$S(t)B \subset B_0 (t \ge t_0);$$

here B_0 and B are bounded sets.

3) When $t > 0$, $S(t)$ is a completely continuous operator.

Therefore, the semigroup operators $S(t)$ exist a compact global attractor A.

Theorem 4.2. [12] *Under the assume of Theorem* 3.1, *equations have global attractor*

$$A = \omega(B_0) = \bigcap_{s \ge 0} \overline{\bigcup_{t \ge s} S(t) B_0},$$

where $B_0 = \left\{(u,v) \in V_2 \times V_1 : \|(u,v)\|^2_{V_2 \times V_1} = \|u\|^2_{V_2} + \|v\|^2_{V_1} \le c(R_0) + c(R_1)\right\}$; B_0 is the bounded absorbing set of $V_2 \times V_1$ and satisfies

(1) $S(t)A = A, t > 0$;

(2) $\lim\limits_{t \to \infty} \operatorname{dist}(S(t)B, A) = 0$, here $B \subset V_2 \times V_1$ and it is a bounded set,

$$\operatorname{dist}(S(t)B, A) = \sup_{x \in B,} \inf_{y \in A} \|S(t)x - y\|_{V_2 \times V_1}.$$

Proof. Under the conditions of Theorem 3.1, it exists the solution semigroup $S(t)$, here $E_1 = V_2 \times V_1, S(t): E_1 \to E_1$.

(1) From Lemma 3.1-Lemma 3.2, we can get that $\forall B \subset V_2 \times V_1$ is a bounded set that includes in the ball $\left\{\|(u,v)\|_{V_2 \times V_1} \le R\right\}$,

$$\|S(t)(u_0, v_0)\|^2_{V_2 \times V_1} = \|u\|^2_{V_2} + \|v\|^2_{V_1} \le \|u_0\|^2_{V_2} + \|v_0\|^2_{V_1} + c \le R^2 + c, \left(t \ge 0, (u_0, v_0) \in B\right).$$

This shows that $S(t)(t \ge 0)$ is uniformly bounded in $V_2 \times V_1$.

(2) Furthermore, for any $(u_0, v_0) \in V_2 \times V_1$, when $t \ge \max\{t_1, t_2\}$, we have

$$\|S(t)(u_0, v_0)\|^2_{V_2 \times V_1} = \|u\|^2_{V_2} + \|v\|^2_{V_1} \le c(R_1) + c(R_2).$$

So we get B_0 is the bounded absorbing set.

(3) Since $V_2 \times V_1 \to V_1 \times H$ is compact embedded, which means that the bounded set in $V_2 \times V_1$ is the compact set in $V_1 \times H$, so the semigroup operator $S(t)$ is completely continuous. ■

Hence, the semigroup operator $S(t)$ exists a compact global attractor A. The proving is completed.

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